State Space Analysis

Dr. Reenu George
Assistant Professor
Electrical and Electronics Engineering Department
Mar Athanasius College of Engineering, Kothamangalam

State Space Approach

- The analysis and design of feedback control systems were carried out using transfer function, together with a variety of graphical tools such as Root locus, Bode plot, Nyquist plot, etc. - Classical or Conventional Control Theory.
- The limitations in the transfer function model and analysis are
    1. Transfer function is defined under zero initial conditions.
    2. Transfer function is applicable to linear time-invariant systems.
    3. Transfer function analysis is restricted to single input and single output systems.
    4. It does not provide information regarding the internal state of the system.
- This led to the development of an approach based on the concept of State- Modern Control Theory, which is a new approach to the analysis and design of complex control systems.

Modern Control Theory Vs Conventional Control Theory

Conventional Control Theory (Transfer function approach)

- Applicable only to linear time-invariant single-input, single-output systems
- With zero initial conditions
- Provide no information regarding the internal state of the system
- Not computer friendly
- Using input-output relationship or transfer function
- Variables represent physical quantities of the system and are measurable.

Modern Control Theory (State space approach)

- Applicable to multiple-input, multiple-output systems, which may be linear or nonlinear, time-invariant or time-varying
- Any initial conditions
- It provides information regarding the internal state of the system
- Easier for analysis using computers
- Any variables of the system can be used
- It is not necessary that the state variables represent physical quantities of the system, but variables that do not represent physical quantities and those that are neither measurable nor observable may be chosen as state variables.

State of a System

- The state of a dynamic system is the smallest set of variables (called state variables) such that, knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.

- The concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State Variables

- The state variables of a dynamic system are the variables making up the smallest set of variables that determines the state of the dynamic system.
- If at least n variables, $x_1, x_2, \ldots\ldots x_n$ are needed to completely describe the behaviour of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.
- Variables that do not represent physical quantities and those that are neither measurable nor observable can be chosen as state variables.
- Physical variables - State variables measured in terms of physical quantities
- Phase variables - State variables which are obtained from one of the system variables and its derivatives, not measurable

State Vector

- If n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of a vector **x**. Such a vector is called a state vector.
- A vector whose elements are state variables

$$\begin{bmatrix} x_1(t) \\ : \\ x_n(t) \end{bmatrix}$$

- A state vector is a vector that determines uniquely the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State Space

- The n-dimensional space whose coordinate axes consist of the $x_1(t)$ axis, $x_2\,(t)$ axis,…… $x_n(t)$ axis, where $x_1(t), x_2(t), \ldots\ldots\ldots x_n\,(t)$ are the state variables, is called a state space.
- "State space" refers to the space whose axes are the state variables.
- The state of the system can be represented as a vector within the space.

State Space Formulation

- In state-space analysis, three types of variables are involved in the modeling of dynamic systems:

Input variables

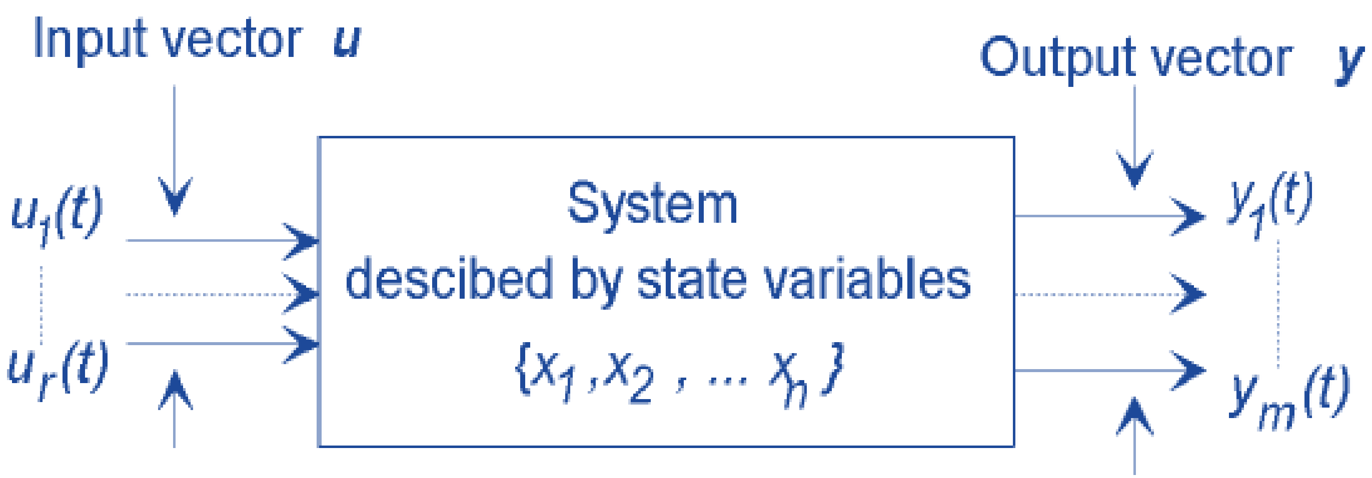
Output variables

State variables

- The state-space representation for a given system is not **unique**, except that the number of state variables is the same for any of the different state-space representations of the same system.
- State space representation - State Equation + Output Equation

State Space Formulation

- Consider a system with r input variables $u_1(t) \dots u_r(t)$, m output variables $y_1(t) \dots y_m(t)$, and n state variables $x_1(t) \dots x_n(t)$



State Space Formulation

$\mathbf{x}(t)$=State vector of order (*n* x 1)

$\mathbf{u}(t)$ =Input vector of order (*r* x 1)

$\mathbf{y}(t)$=Output vector of order (*m*x*1*)

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ : \\ : \\ u_r(t) \end{bmatrix} \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ : \\ : \\ x_n(t) \end{bmatrix} \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ : \\ : \\ y_m(t) \end{bmatrix}$$

State Equation

- State equation of a system is a function of state variables and input variables.
- Fromulated by n number of first order differential equation which relates state variables and input variables.
- Note $\frac{dx}{dt} = \dot{x}(t) = \dot{x}$

$$\dot{x}_1(t) = f_1(x_1, x_2, \dots \dots .. x_n ; u_1, u_2, \dots \dots .. u_r)$$

$$\dot{x}_2(t) = f_2(x_1, x_2, \dots \dots .. x_n ; u_1, u_2, \dots \dots .. u_r)$$

$$\dot{x}_3(t) = f_3(x_1, x_2, \dots \dots .. x_n ; u_1, u_2, \dots \dots .. u_r)$$

$$\vdots$$

$$\dot{x}_n(t) = f_n(x_1, x_2, \dots \dots .. x_n ; u_1, u_2, \dots \dots .. u_r)$$

In vector notation, $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$

Output Equation

- Output equation of a system - output expressed as a function of state variables and input variables.

$$y_1(t) = g_1(x_1, x_2, \ldots \ldots .. x_n; u_1, u_2, \ldots \ldots .. u_r)$$

$$y_2(t) = g_2(x_1, x_2, \ldots \ldots .. x_n; u_1, u_2, \ldots \ldots .. u_r)$$

$$y_3(t) = g_3(x_1, x_2, \ldots \ldots .. x_n; u_1, u_2, \ldots \ldots .. u_r)$$

$$\vdots$$

$$y_m(t) = g_m(x_1, x_2, \ldots \ldots .. x_n; u_1, u_2, \ldots \ldots .. u_r)$$

In vector notation, $\boldsymbol{y}(t) = g(\mathbf{x}(t), \mathbf{u}(t))$

State Equation Linear Time-Invariant System

- The first order differential equation can be expressed as a linear combination of state variables and inputs.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots b_{1r}u_r$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots b_{2r}u_r$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots b_{nr}u_r$$

In matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & . & b_{1r} \\ b_{21} & b_{22} & . & b_{2r} \\ . & & & \\ . & & & \\ b_{n1} & b_{n2} & . & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_r \end{bmatrix}$$

State Equation Linear Time-Invariant System

Matrix equation can be written as,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ : \\ : \\ x_n(t) \end{bmatrix} \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ : \\ : \\ u_r(t) \end{bmatrix}$$

Where,

$\mathbf{x}(t)$=State vector of order (n x 1)

$\mathbf{u}(t)$ =Input vector of order (r x 1)

A=System matrix of order (n x n)

B=Input matrix of order (n x r)

State equation, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

Output Equation Linear Time-Invariant System

- For a linear time-invariant system, the output at any time can be expressed as a linear combination of state variables and input variables.

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots d_{1r}u_r$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots d_{2r}u_r$$

$$\vdots$$

$$\vdots$$

$$y_m = c_{m1}x_1 + c_{m2}x_2 + \cdots c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \cdots d_{mr}u_r$$

In matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & . & c_{1n} \\ c_{21} & c_{22} & . & . & c_{2n} \\ . & & & & \\ . & & & & \\ c_{m1} & c_{m2} & . & . & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & . & d_{1r} \\ d_{21} & d_{22} & . & d_{2r} \\ . & & & \\ . & & & \\ d_{m1} & d_{m2} & . & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_r \end{bmatrix}$$

Output Equation Linear Time-Invariant System

Matrix equation can be written as,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ : \\ : \\ y_m(t) \end{bmatrix}$$

Where,

$\mathbf{y}(t)$=Output vector of order(mx1)

C=Output matrix of order (mxn)

D=Transmission matrix of order (mxr)

Output equation, $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$

State Model for Continuous Time Systems

- For a linear time-invariant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- For a linear time-variant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

- For a nonlinear time-invariant system,

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$$
$$\mathbf{y}(t) = \mathrm{g}(\mathbf{x}(t), \mathbf{u}(t))$$

- For a nonlinear time-variant system,

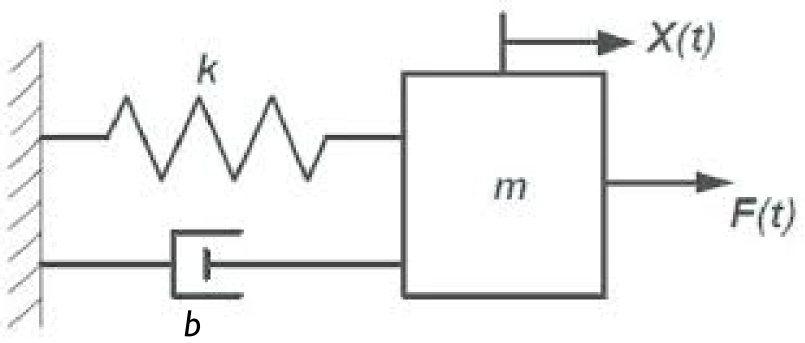
$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$$
$$\mathbf{y}(t) = \mathrm{g}(\mathbf{x}(\mathrm{t}), \mathbf{u}(\mathrm{t}), t)$$

Example: Consider a mass spring damper system whose governing equation is given by,

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

Let the state variables be $x_1(t) = x(t) \qquad x_2(t) = \dot{x}(t)$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{b}{m}x_2(t) + \frac{1}{m}u(t)$$
$$y(t) = x_1(t)$$



$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k/m & -b/m\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

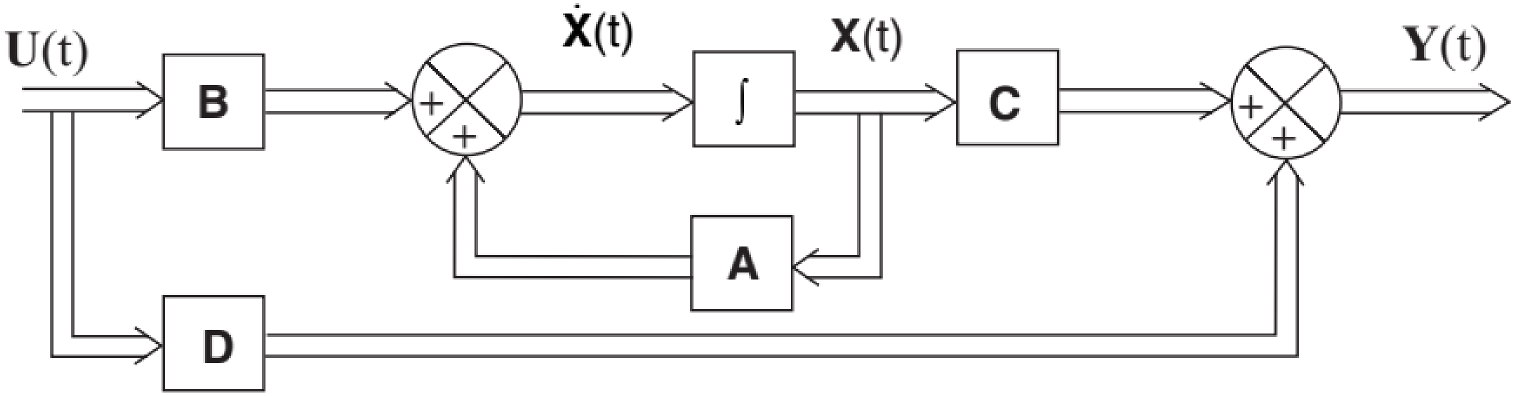
$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

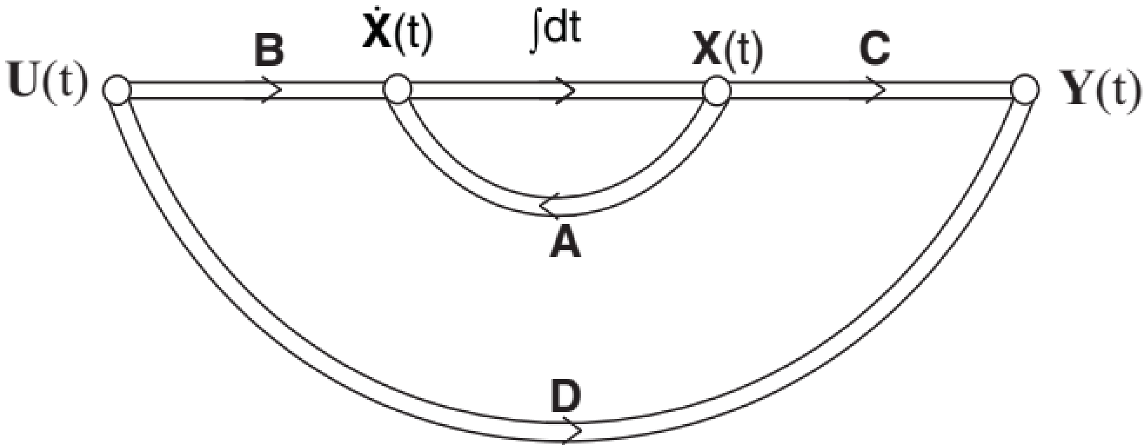
State Diagram

- The pictorial representation of the state model of the system is called State diagram.
- The state diagram of the system can be either in Block Diagram form or in Signal flow graph form.
- The state diagram describes the relationships among the state variables and provides physical interpretations of the state variables.
- The time domain state diagram may be obtained directly from the differential equation governing the system and this diagram can be used for simulation of the system in analog computers.
- The s-domain state diagram can be obtained from the transfer function of the system.

State Diagram



Block diagram of state model.



Signal flow graph of state model.

For a linear time-invariant system, state model is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

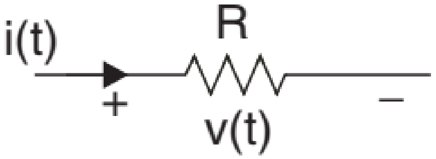
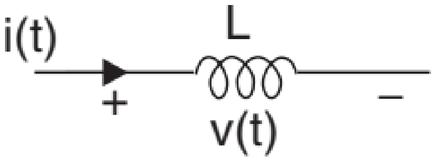
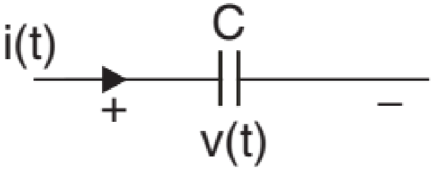
- The dynamic system must involve elements that memorize the values
- Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system.
- Thus the outputs of integrators serve as state variables.

State-Space Representation using Physical Variables

- In state-space modeling of systems, the choice of state variables is arbitrary. One of the possible choice of state variables is the physical variables.
- The physical variables of electrical systems are Current or Voltage in the R, L, and C elements.
- The physical variables of mechanical systems are Displacement, Velocity, and Acceleration.
- The advantages of choosing the physical variables (or quantities) of the system as state variables are the following,
  1. The state variables can be utilized for the purpose of feedback.
  2. The implementation of design with state variable feedback becomes straightforward.
  3. The solution of the state equation gives time variation of variables which have direct relevance to the physical system
- The drawback in choosing the physical quantities as state variables is that the solution of state equation may become a difficult task.
- In state space modeling using physical variables, the state equations are obtained from the differential equations governing the system, which are obtained from a basic model of the system which is developed using the fundamental elements of the system.

State-Space Representation - Electrical System

- The basic model of an electrical system can be obtained by using the fundamental elements Resistor, Capacitor and Inductor.
- Using these elements the electrical network or equivalent circuit of the system is drawn.

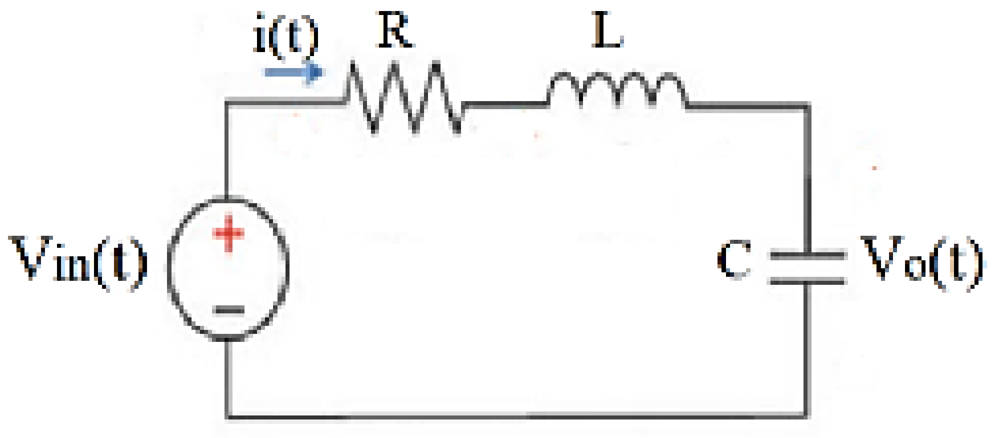
Element	Voltage across the element	Current through the element
	$v(t) = Ri(t)$	$i(t) = \frac{v(t)}{R}$
	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int v(t)\,dt$
	$v(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

State-Space Representation - Electrical System

- The differential equations governing the electrical systems can be formed - Kirchhoff’s Current Law equations or Kirchhoff's Voltage Law
- A minimal number of state variables are chosen for obtaining the state model of the system.
- The best choice of state variables in electrical systems are currents and voltages in energy storage elements - inductance and capacitance.
- The physical variables in the differential equations are replaced by state variables, and the equations are rearranged as first-order differential equations.
- This set of first-order equations constitutes the state equation of the system.
- The inputs to the system are exciting voltage sources or current sources.
- The outputs in electrical systems are usually voltages or currents in energy-dissipating elements – resistance.
- In general, the output variables can be any voltage or current in the network.

State-Space Representation-Electrical System Example

Obtain the state model of the RLC electrical circuit as shown in figure.



- Let us choose the current through the inductances and voltage across the capacitor as state variables (two energy storing elements).
- Let two state variables be x_1 and x_2

First state variable x_1 = current through the inductor $i(t)$

Second state variable x_2 = voltage across capacitor $V_0(t)$

State-space Representation-Electrical System Example

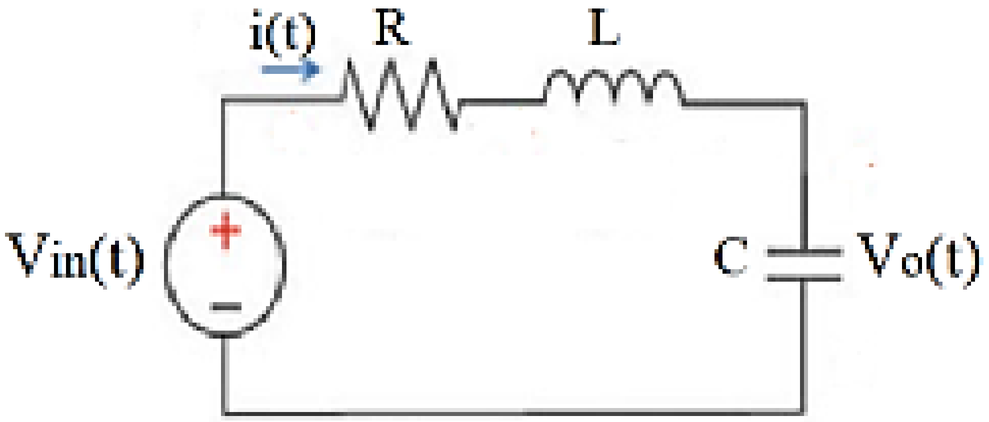
- To form the differential equation, we can write either Kirchhoff's voltage law or current law

Using KVL, $V_{in}(t) = R\, i(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + V_0(t)$ ……….. (1)

- Voltage across the capacitor is given by

$$V_0(t) = \frac{1}{C}\int i(t)\,dt$$

On taking the derivative, $\frac{dV_0}{dt} = \frac{1}{C}\, i(t)$ …. (2)



- Substituting for state variables,

(1), and (2) becomes $u = Rx_1 + L\dot{x}_1 + x_2$

$$\dot{x}_2 = \frac{1}{C}x_1$$

State-Space Representation-Electrical System Example

- Rearranging $u = Rx_1 + L\dot{x}_1 + x_2$

$$\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u$$

$$\dot{x}_2 = \frac{1}{C}x_1$$

State equation

- State equation in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-R}{L} & \frac{-1}{L}\\ \frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}u$$

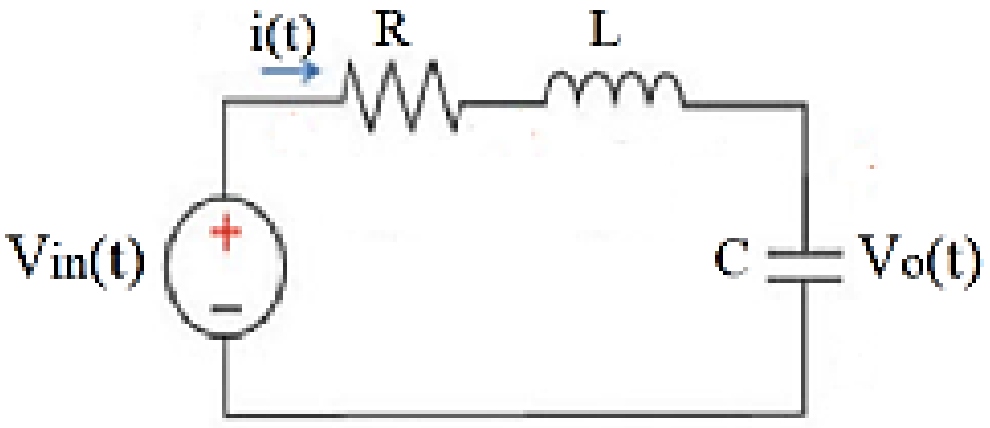
- Out put equation , voltage across capacitor is $y = x_2$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

State-Space Representation-Electrical System Example

- The same problem with different state variables

Using KVL, $V_{in}(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt \ldots\ldots\ldots..(1)$



- In terms of charge $q(t)$ (1) becomes

$$V_{in}(t) = R\frac{dq(t)}{dt} + L\frac{d^2q(t)}{dt^2} + \frac{1}{C}q(t) \ldots\ldots\ldots. \quad (2)$$

- Where $i(t) = \frac{dq(t)}{dt}$

State-Space Representation-Electrical System Example

- Let two state variables be x_1 and x_2

$$x_1 = q(t), \quad x_2 = \frac{dq(t)}{dt}$$

Now, we have $\dot{x}_1 = \frac{dq(t)}{dt} = x_2$

$$\dot{x}_1 = x_2$$

(1) in terms of state variables

$$V_{in}(t) = R\frac{dq(t)}{dt} + L\frac{d^2q(t)}{dt^2} + \frac{1}{C}q(t)$$

we have $\frac{d^2q(t)}{dt^2} = \dot{x}_2$

$$u = Rx_2 + L\dot{x}_2 + \frac{1}{C}x_1$$

Rearranging,

$$\dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u$$

State-Space Representation-Electrical System Example

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u$$

State equation

- State equation in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

- Out put equation, current as output variable, $y = x_2$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State Space Representation by Phase Variables Forms, Controllable and Observable Canonical Form

State-Space Representation-Phase Variables

- The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives.
- Usually, the variable used is the system output, and the remaining state variables are the derivatives of the output.
- The state model using phase variables can be easily determined if the system model is already known in the differential equation or transfer function form.

State-Space Representation-Phase Variables

- Consider the following n^{th} order linear differential equation relating the output $y(t)$ to the input $u(t)$ of a system,

$$\overset{n.}{y} + a_1 \overset{(n-1).}{y} + a_2 \overset{(n-2).}{y} + \ldots\ldots\ldots + a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = b\,u$$

By choosing the output **y** and their derivatives as state variables, we get,

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$x_3 = \ddot{y}$$

$$\vdots$$

$$x_n = \overset{(n-1).}{y} \quad ; \quad \therefore \dot{x}_n = \overset{n.}{y}$$

On substituting the state variables in the differential equation governing the system

State-Space Representation-Phase Variables

- We get,

$$\dot{x}_n + a_1 x_n + a_2 x_{n-1} + \dots\dots\dots + a_{n-2} x_3 + a_{n-1} x_2 + a_n x_1 = b\,u$$

$$\therefore \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \dots\dots\dots\dots\dots - a_2 x_{n-1} - a_1 x_n + b\,u$$

The state equations of the system are

$$\dot{x}_1 = x_2$$

$$x_1 = y$$

$$x_2 = \dot{y}$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \dots\dots\dots - a_2 x_{n-1} - a_1 x_n + b\,u$$

On arranging the above equations in the matrix form we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} [u]$$

Phase Variable form

or $\dot{\mathbf{X}} = \mathbf{A}\,\mathbf{X} + \mathbf{B}\,\mathbf{U}$

- The matrix A (system matrix) has a very special form.
- It has all 1's in the upper off-diagonal. Its last row is comprised of the negative of the coefficients of the original differential equation and all other elements are zero.
- This form of matrix A is known as the Bush form (or) Companion form.
- The B matrix has all its elements except the last element, zero.

State-Space Representation-Phase Variables

- The output being $y = x_1$, the output equation is given by

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots\cdots\cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$\text{(or) } \mathbf{Y} = \mathbf{CX}$$

- The advantage of using phase variables for state space modeling is that the system state model can be written directly by inspection from the differential equation governing the system.

State-Space Representation-Phase Variables Example 1

Q. Construct a state model for a system characterized by the differential equation,

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$$

Give the block diagram representation of the state model.

SOLUTION

Let us choose y and their derivatives as state variables. The system is governed by third order differential equation and so the number of state variables are three.

The state variables x_1, x_2 and x_3 are related to phase variables as follows

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1$$

$$x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$$

Put $y = x_1$, $\frac{dy}{dt} = x_2$ and $\frac{d^2y}{dt^2} = x_3$ and $\frac{d^3y}{dt^3} = \dot{x}_3$ in the given equation,

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$$

$$\therefore \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\text{or} \quad \dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

On arranging the state equations in the matrix form we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u]$$

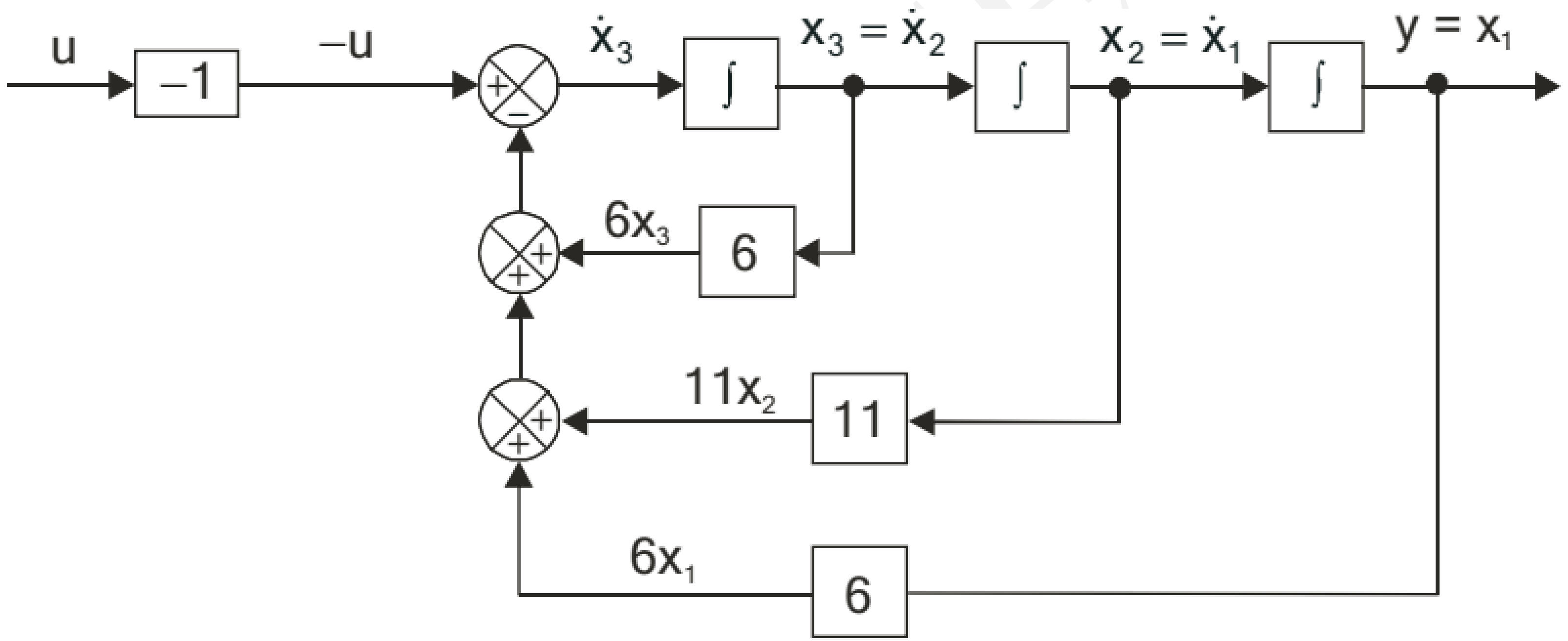
Here, y = output

But, $y = x_1$

$$\therefore \text{The output equation is, } y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The state equation and output equation, constitutes the state model of the system.

The block diagram form of the state diagram of the system



State-space Representation-Phase Variables Example 2

Q. Obtain the state model of the system whose transfer function is given as,

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$$

SOLUTION

Given that, $\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$

On cross multiplying the equation

$$Y(s)[s^3 + 4s^2 + 2s + 1] = 10\ U(s)$$

$$s^3Y(s) + 4s^2Y(s) + 2s\ Y(s) + Y(s) = 10\ U(s)$$

On taking inverse Laplace transform of equation

$$\dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u$$

Let us define state variables as follows,

$$x_1 = y \quad ; \quad x_2 = \dot{y} \quad ; \quad x_3 = \ddot{y}$$

Put $\dddot{y} = \dot{x}_3 \quad ; \quad \ddot{y} = x_3 \quad ; \quad \dot{y} = x_2$ and $y = x_1$ in the equation

$$\therefore \dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$$

$$\text{or } \dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$$

The state equations are

$$\dot{x}_1 = x_2 \quad ; \quad \dot{x}_2 = x_3 \quad ; \quad \dot{x}_3 = -x_1 - 2x_2 - 4x_3 + 10u$$

The output equation is $y = x_1$

The state model in the matrix form is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u] \quad ; \quad y = [1 \ \ 0 \ \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-Space Representation - Controllable Canonical Form (CCF) or Phase Variable Canonical Form

- Consider the following n^{th} order linear differential equation relating the output $y(t)$ to the input $u(t)$ of a system,

$$\overset{n.}{y} + a_1 \overset{(n-1).}{y} + \dots\dots\dots\dots + a_{n-1}\dot{y} + a_n y = b_0 \overset{m.}{u} + b_1 \overset{(m-1).}{u} + \dots\dots + b_{m-1}\dot{u} + b_m u$$

let $n = m = 3$

$$\therefore \dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

On taking Laplace transform of equation with zero initial conditions we get,

$$s^3Y(s) + a_1s^2Y(s) + a_2sY(s) + a_3Y(s) = b_0s^3U(s) + b_1s^2U(s) + b_2sU(s) + b_3U(s)$$

$$(s^3 + a_1s^2 + a_2s + a_3)\, Y(s) = (b_0s^3 + b_1s^2 + b_2s + b_3)U(s)$$

State-Space Representation - Controllable Canonical Form (CCF) or Phase Variable Canonical Form

$$\therefore \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Let $\frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)} \cdot \frac{Y(s)}{X_1(s)}$

where, $\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$

and $\frac{Y(s)}{X_1(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$

On cross multiplying the equation

$$X_1(s)[s^3 + a_1 s^2 + a_2 s + a_3] = U(s)$$

$$s^3 X_1(s) + a_1 s^2 X_1(s) + a_2 s X_1(s) + a_3 X_1(s) = U(s)$$

On taking inverse laplace transform of equation

$$\dddot{x}_1 + a_1 \ddot{x}_1 + a_2 \dot{x}_1 + a_3 x_1 = u$$

State-Space Representation - Controllable Canonical Form (CCF) or Phase Variable Canonical Form

Let the state variable be, x_1, x_2 and x_3

where, $x_2 = \dot{x}_1$

and $x_3 = \ddot{x}_1 = \dot{x}_2 \quad ; \quad \therefore \dot{x}_3 = \dddot{x}_1$

On substituting the state variables in equation $\dddot{x}_1 + a_1\ddot{x}_1 + a_2\dot{x}_1 + a_3x_1 = u$

$$\dot{x}_3 + a_1x_3 + a_2x_2 + a_3x_1 = u$$

$$\therefore \dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + u$$

State-Space Representation - Controllable Canonical Form (CCF) or Phase Variable Canonical Form

The state equations are,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u$$

We have $$\frac{Y(s)}{X_1(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$

On cross multiplying the equation

$$Y(s) = b_0 s^3 X_1(s) + b_1 s^2 X_1(s) + b_2 s X_1(s) + b_3 X_1(s)$$

On taking inverse Laplace transform of equation

$$y = b_0 \dddot{x}_1 + b_1 \ddot{x}_1 + b_2 \dot{x}_1 + b_3 x_1$$

State-Space Representation - Controllable Canonical Form (CCF) or Phase Variable Canonical Form

On substituting the state variables in equation

$$y = b_0\dot{x}_3 + b_1x_3 + b_2x_2 + b_3x_1$$

Put $\dot{x}_3 = -a_3x_1 - a_2x_2 - a_1x_3 + u$ in equation

$$\therefore y = b_0(-a_3x_1 - a_2x_2 - a_1x_3 + u) + b_1x_3 + b_2x_2 + b_3x_1$$

$$y = (b_3 - a_3b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_1 - a_1b_0)x_3 + b_0u$$

The equation is the output equation.

State-Space Representation - Controllable Canonical Form (CCF) or Phase Variable Canonical Form

On arranging the state equations and output equations in the matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

$$y = [(b_3 - a_3 b_0) \ \ (b_2 - a_2 b_0) \ (b_1 - a_1 b_0)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] \ u$$

Generalized for an nth order differential equation-
Controllable Canonical Form or Phase Variable Canonical Form

$$\overset{n.}{y} + a_1 \overset{(n-1).}{y} + \dots\dots\dots\dots\dots + a_{n-1}\dot{y} + a_n y = b_0 \overset{m.}{u} + b_1 \overset{(m-1).}{u} + \dots\dots + b_{m-1}\dot{u} + b_m u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots\dots & 0 \\ 0 & 0 & 1 & \dots\dots & 0 \\ \vdots & \vdots & \vdots & \dots\dots & \vdots \\ 0 & 0 & 0 & \dots\dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots\dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [u]$$

$$y = \left[(b_n - a_n b_0)(b_{n-1} - a_{n-1} b_0)\dots\dots(b_2 - a_2 b_0)(b_1 - a_1 b_0)\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

State-space Representation- Controllable Canonical Form: Example

Q. Obtain the state model of the system whose transfer function is given as,

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$$

Let $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} \frac{X_1(s)}{U(s)}$ Where, $\frac{Y(s)}{X_1(s)} = 10$ $\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + 4s^2 + 2s + 1}$

Take, $\frac{X_1(s)}{U(s)} = \frac{1}{s^3+4s^2+2s+1}$ On cross multiplying,

$$X_1(s)(s^3 + 4s^2 + 2s + 1) = U(s)$$

$$s^3X_1(s) + 4s^2X_1(s) + 2sX_1(s) + X_1(s) = U(s)$$

$$s^3X_1(s) + 4s^2X_1(s) + 2sX_1(s) + X_1(s) = U(s)$$

On taking inverse Laplace Transform

$$\dddot{x}_1 + 4\ddot{x}_1 + 2\dot{x}_1 + x_1 = u$$

Let the state variables be x_1, x_2 and x_3

Where $x_2 = \dot{x}_1$

$$x_3 = \ddot{x}_1 = \dot{x}_2$$

$$\dot{x}_3 = \dddot{x}_1$$

On subsisting the state variables,

$$\dot{x}_3 + 4x_3 + 2\,x_2 + x_1 = u$$

$$\dot{x}_3 = -4x_3 - 2\,x_2 - x_1 + u$$

The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -x_1 - 2x_2 - 4x_3 + u$$

In matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Output Equation

$$\frac{Y(s)}{X_1(s)} = 10, \qquad Y(s) = 10X_1(s)$$

On taking inverse Laplace

$$y = 10x_1$$

In matrix form,

$$y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-Space Representation - Observable Canonical Form (OCF)

- Consider the following n^{th} order linear differential equation relating the output $y(t)$ to the input $u(t)$ of a system,

$$\overset{n.}{y} + a_1 \overset{(n-1).}{y} + \ldots\ldots\ldots\ldots\ldots + a_{n-1}\dot{y} + a_n y = b_0 \overset{m.}{u} + b_1 \overset{(m-1).}{u} + \ldots\ldots + b_{m-1}\dot{u} + b_m u$$

let n = m = 3

$$\therefore \dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

On taking Laplace transform of equation (4.14) with zero initial conditions we get,

$$s^3Y(s) + a_1s^2Y(s) + a_2sY(s) + a_3Y(s) = b_0s^3U(s) + b_1s^2U(s) + b_2sU(s) + b_3U(s)$$

$$(s^3 + a_1s^2 + a_2s + a_3)\,Y(s) = (b_0s^3 + b_1s^2 + b_2s + b_3)U(s)$$

State-Space Representation - Observable Canonical Form (OCF)

$$\therefore \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} = \frac{s^3\left(b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}\right)}{s^3\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right)} = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right)}$$

From the Mason's gain formula, the transfer function of the system is given by,

$$\mathbf{T(s)} = \frac{\mathbf{1}}{\Delta} \sum_{K} \mathbf{P}_K \Delta_K$$

where, P_K = path gain of K^{th} forward path.

Δ = 1– (sum of loop gain of all individual loops)

+ (sum of gain products of all possible combinations of two non-touching loops) – ·········

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path.

State-Space Representation - Observable Canonical Form (OCF)

The transfer function of a system with four forward paths and with three feedback loops (touching each other) is given by,

$$T(s) = \frac{P_1 + P_2 + P_3 + P_4}{1 - (P_{11} + P_{12} + P_{13})}$$

On comparing equation above transfer function Y(s)/U(s)

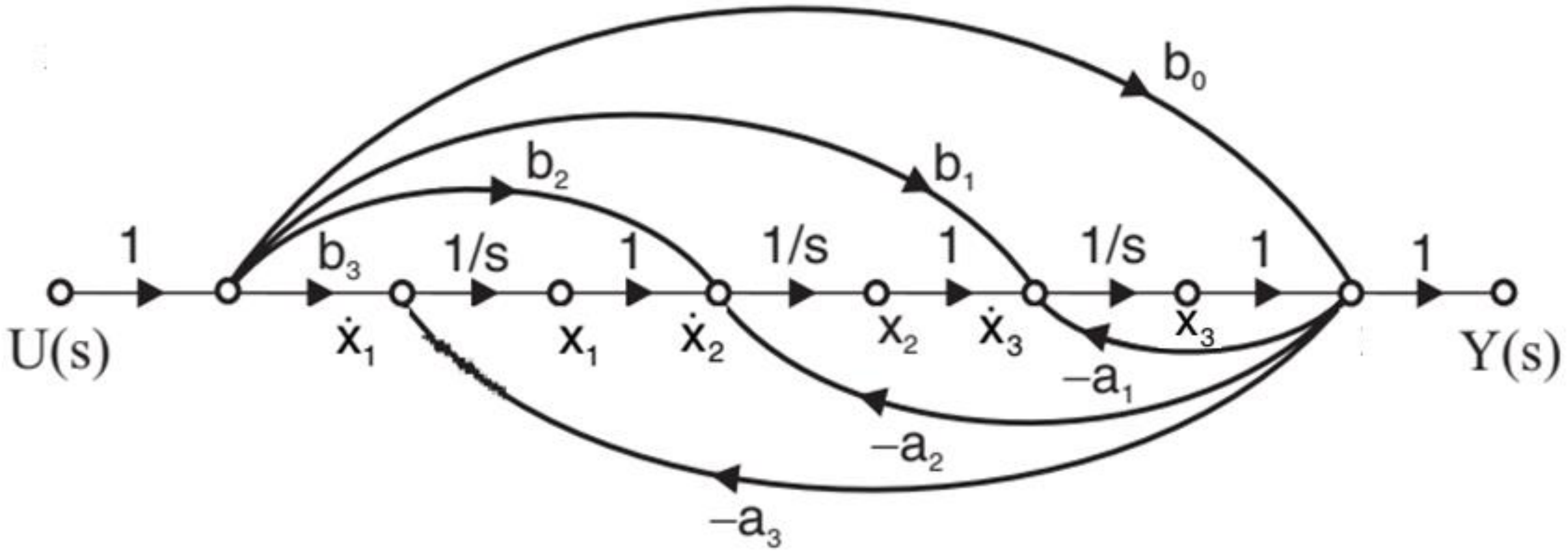
$$P_1 = b_0 \quad ; \quad P_2 = \frac{b_1}{s} \quad ; \quad P_3 = \frac{b_2}{s^2} \quad \text{and} \quad P_4 = \frac{b_3}{s^3}$$

$$P_{11} = -\frac{a_1}{s}; \quad P_{12} = -\frac{a_2}{s^2} \quad \text{and} \quad P_{13} = \frac{a_3}{s^3}$$

State-Space Representation - Observable Canonical Form (OCF)

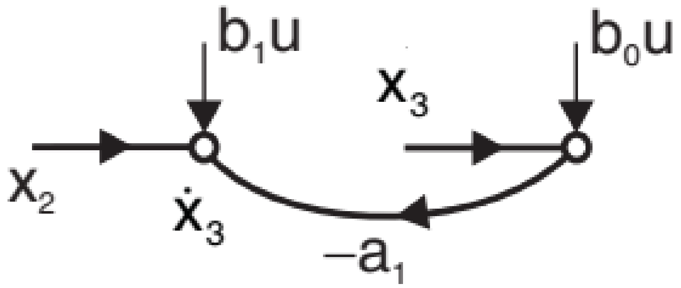
- Hence for the system represented by the transfer function, a signal flow graph can be constructed.
- The signal flow is constructed such that all $\Delta_k = 1$, and all loops are touching loops.
- Let us assign state variables at the output of each integrator in the signal flow graph.
- Hence at the input of each integrator, the first derivative of the state variable will be available.
- The state equations are formed by summing all the incoming signals to the nodes, whose values correspond to the first derivative of state variables

State-Space Representation - Observable Canonical Form (OCF)

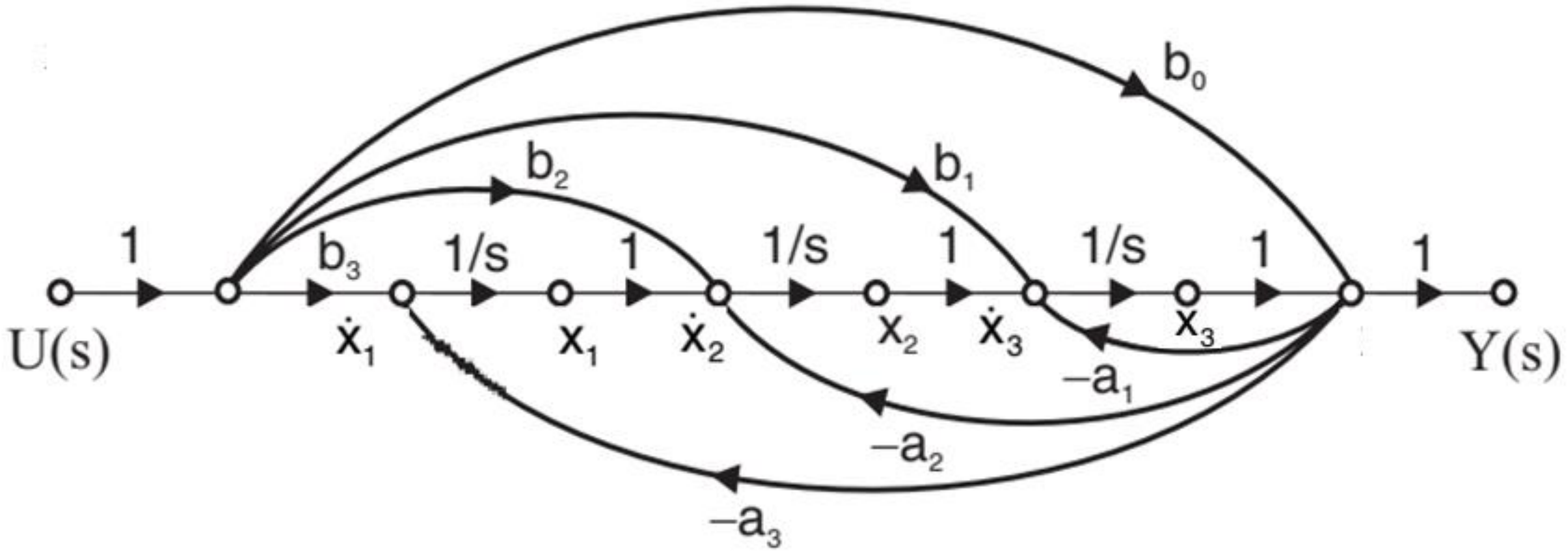


By summing up the incoming signals to node

- $\dot{x}_3 = x_2 - a_1(b_0u + x_3) + b_1u$
- $\dot{x}_3 = x_2 - a_1x_3 - a_1b_0u + b_1u$
- $\dot{x}_3 = x_2 - a_1x_3 + (b_1 - a_1b_0)u$



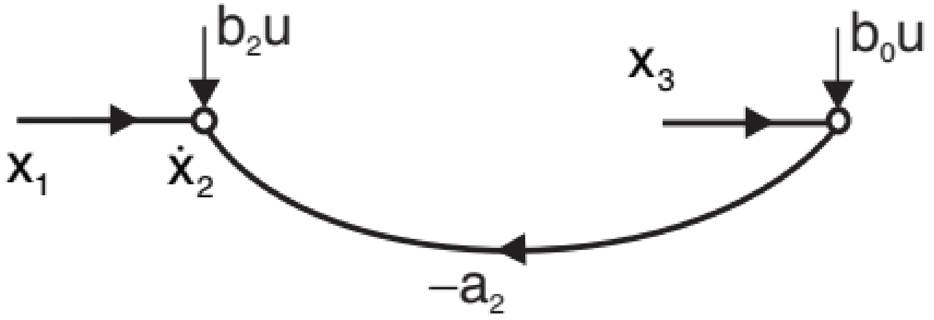
State-Space Representation - Observable Canonical Form (OCF)

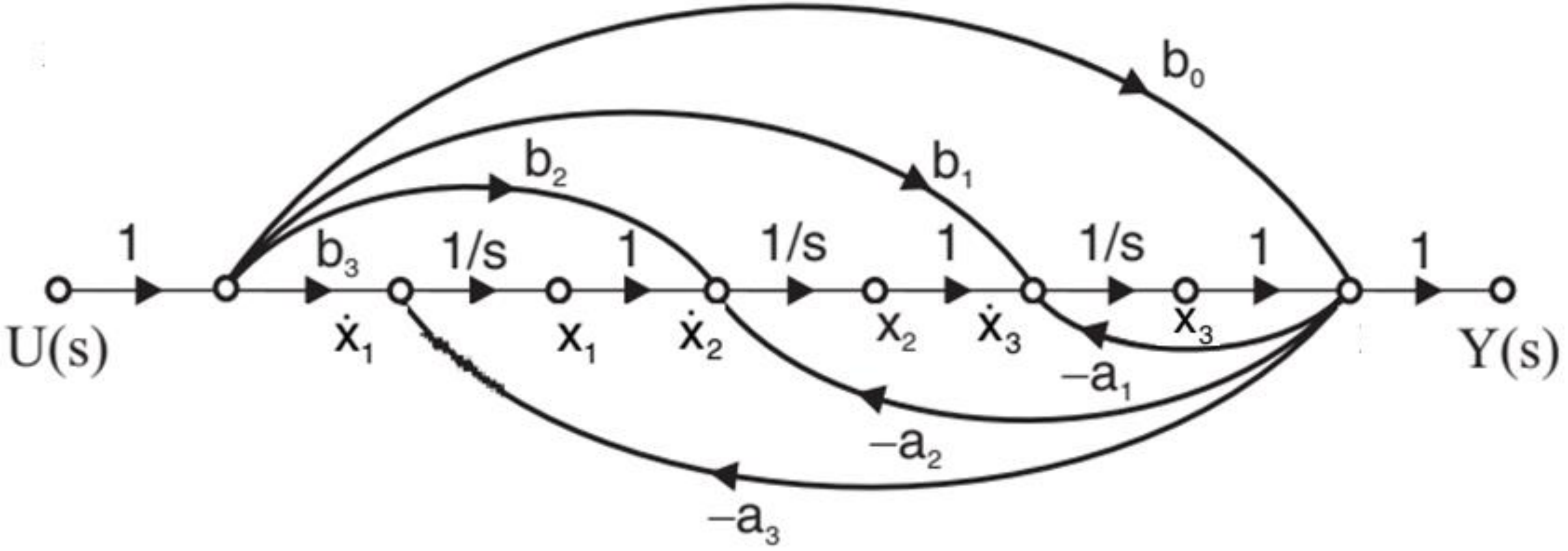


By summing up the incoming signals to node

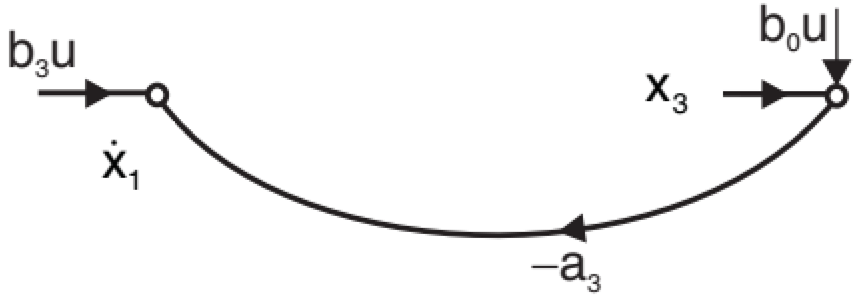
$$\dot{x}_2 = x_1 - a_2(b_0u + x_3) + b_2u$$

$$\dot{x}_2 = x_1 - a_2x_3 + (b_2 - a_2b_0)u$$





By summing up the incoming signals to node



$$\dot{x}_1 = -a_3(b_0 u + x_3) + b_3 u$$

$$\dot{x}_1 = -a_3 x_3 + (b_3 - a_3 b_0)u$$

The output equation is given by the sum of incoming signals to output node.

$$y = x_3 + b_0 u$$

State-Space Representation - Observable Canonical Form (OCF)

- $\dot{x}_1 = -a_3x_3 + (b_3 - a_3b_0)u$
- $\dot{x}_2 = x_1 - a_2x_3 + (b_2 - a_2b_0)u$
- $\dot{x}_3 = x_2 - a_1x_3 + (b_1 - a_1b_0)u$
- On arranging the state equations in the matrix form, we get,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_3 - a_3b_0 \\ b_2 - a_2b_0 \\ b_1 - a_1b_0 \end{bmatrix} u$$

- Output Equation

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0u$$

Generalized for an nth Order Differential equation- Observable Canonical Form

$$\overset{n.}{y} + a_1 \overset{(n-1).}{y} + \dots\dots\dots\dots\dots + a_{n-1}\dot{y} + a_n y = b_0 \overset{m.}{u} + b_1 \overset{(m-1).}{u} + \dots\dots + b_{m-1}\dot{u} + b_m u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Generalized for an nth Order Differential Equation- Relation Between Observable- Controllable Canonical Form (OCF - CCF)

$$
\begin{aligned}
A_{obs} &= A_{cont}^T \\
B_{obs} &= C_{cont}^T \\
C_{obs} &= B_{cont}^T \\
D_{obs} &= D_{cont}
\end{aligned}
$$

- The state space model can be directly formed by inspection from the differential equations governing the system.
- The phase variables provide a link between the transfer function design approach and the time domain approach.
- The phase variables are not physical variables of the system and therefore are not available for measurement and control purposes.

State-Space Representation-Observable Canonical Form Example

Q. Obtain the state model of the system whose transfer function is given as,

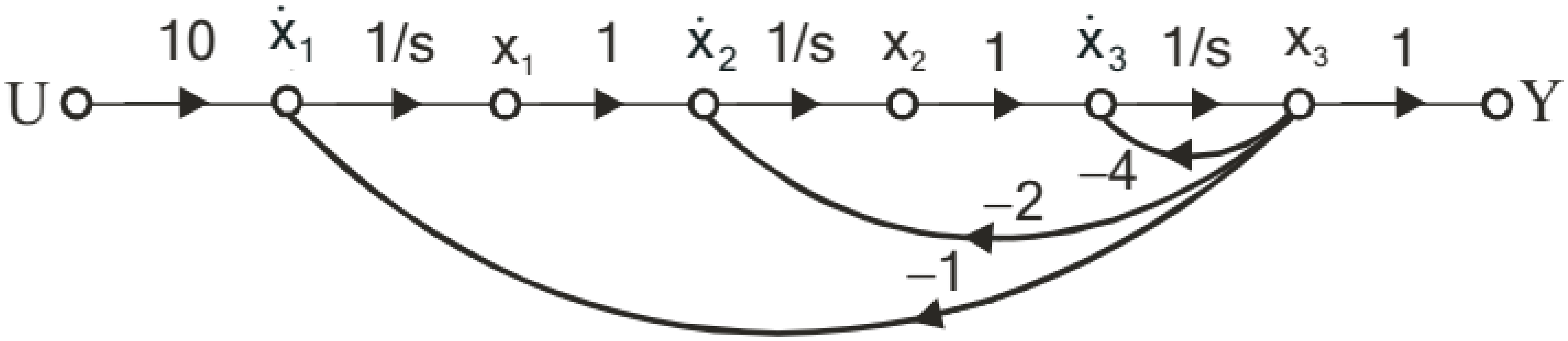
$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1}$$

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 4s^2 + 2s + 1} = \frac{10}{s^3\left(1 + \frac{4}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right)}$$

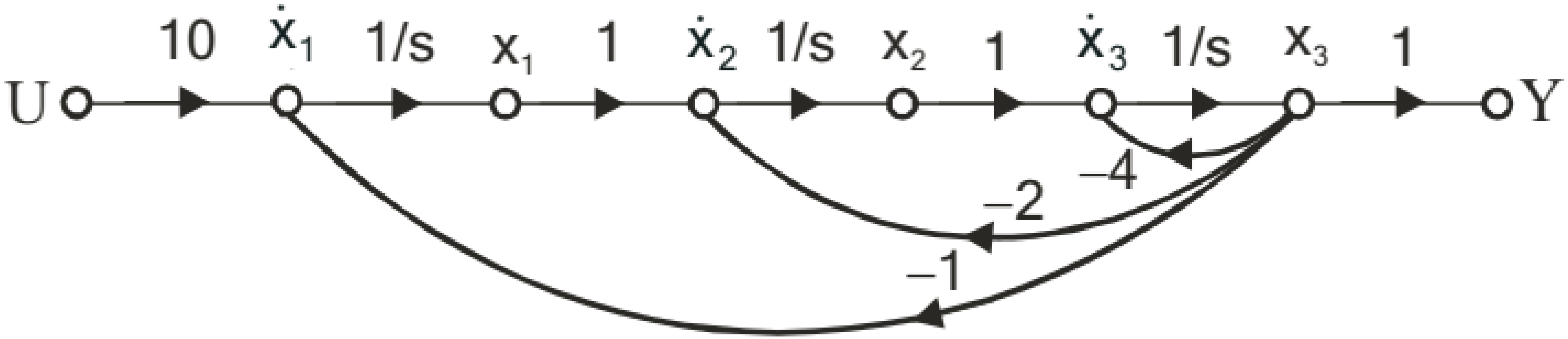
$$= \frac{10/s^3}{1 - \left(-\frac{4}{s} - \frac{2}{s^2} - \frac{1}{s^3}\right)}$$

- The signal flow graph for the above transfer function can be constructed as with a single forward path consisting of three integrators and with path gain $10/s^3$.
- The graph will have three individual loops with loop gains $-4/s$, $-2/s^2$, and $-1/s^3$.

State-Space Representation-Observable Canonical Form Example

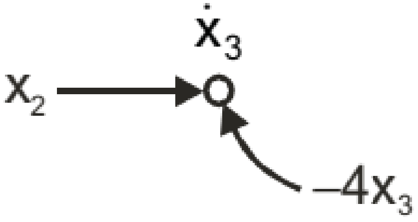


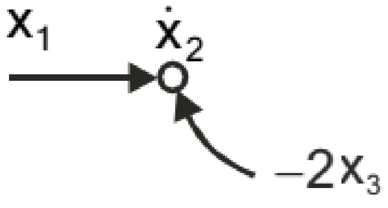
- Assign state variables at the output of the integrator (1/s).
- The state equations are obtained by summing the incoming signals to the input of the integrators and equating them to the corresponding first derivative of the state variable.



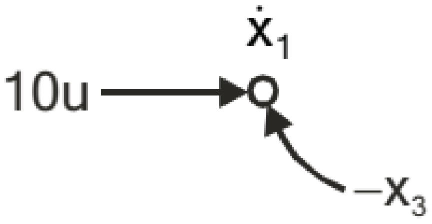
The state equations are

$$\dot{x}_1 = -x_3 + 10u$$
$$\dot{x}_2 = x_1 - 2x_3$$
$$\dot{x}_3 = x_2 - 4x_3$$





The output equation, $y = x_3$



- On arranging the state equations in the matrix form, we get,
- State equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

- Output Equation

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

State-Space Representation - From Block Diagram

Construct a state model for a system characterized by the differential equation,

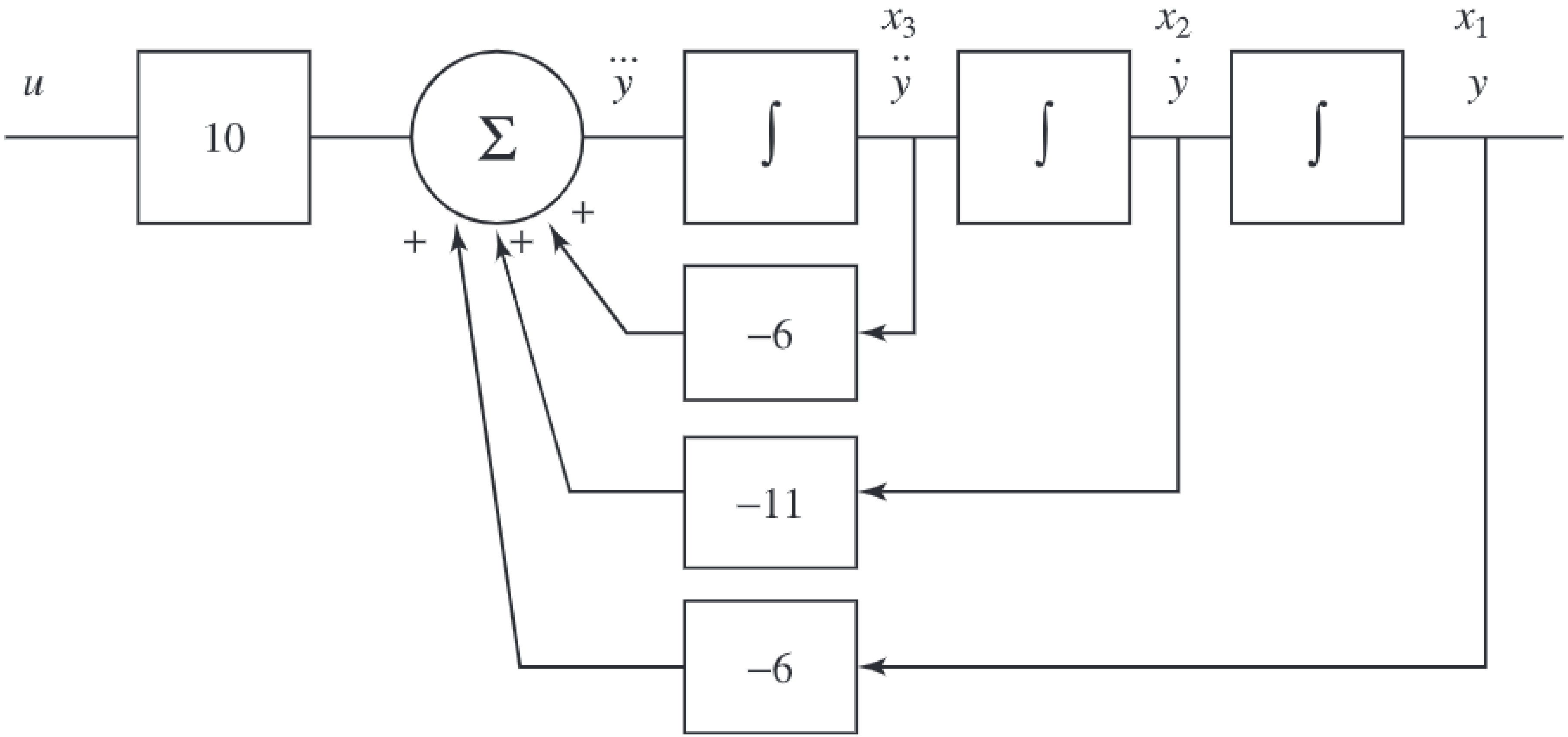
$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 10u.$$

Assuming that the highest derivative is available for integration, lower order derivatives can be generated by successive integration and these can be summed up after proper weighting

The highest derivative is obtained by feeding back the lower order derivatives with proper scaling and then adding to the input term.

$$\dddot{y} = -6\ddot{y} - 11\dot{y} - 6y + 10u$$

State-Space Representation - From Block Diagram



The state variables are chosen as shown, with x_1, x_2, x_3 being identified in reverse order from the output y of the system. The state and output equations can now be written by inspection from the diagram:

State-Space Representation - From Block Diagram

$\dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3$ and $\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 10u$ and output $y = x_1$

In matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

State Space Representation Diagonal and Jordan Canonical Form

State-Space Representation - Diagonal Canonical Form

- In diagonal canonical form (canonical or normal form) of state model, the system matrix A will be a diagonal matrix.
- The elements on the diagonal are the poles of the transfer function of the system.

By partial fraction expansion, the transfer function Y(s)/U(s) of the n^{th} order system can be expressed as

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s+\lambda_1} + \frac{C_2}{s+\lambda_2} + \ldots\ldots\ldots + \frac{C_n}{s+\lambda_n}$$

where $C_1, C_2, C_3, \ldots\ldots\ldots C_n$ are residues and $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are roots of denominator polynomial (or poles of the system).

State-Space Representation - Diagonal Canonical Form

The equation above can be rearranged as

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s\left(1 + \frac{\lambda_1}{s}\right)} + \frac{C_2}{s\left(1 + \frac{\lambda_2}{s}\right)} + \ldots\ldots + \frac{C_n}{s\left(1 + \frac{\lambda_n}{s}\right)}$$

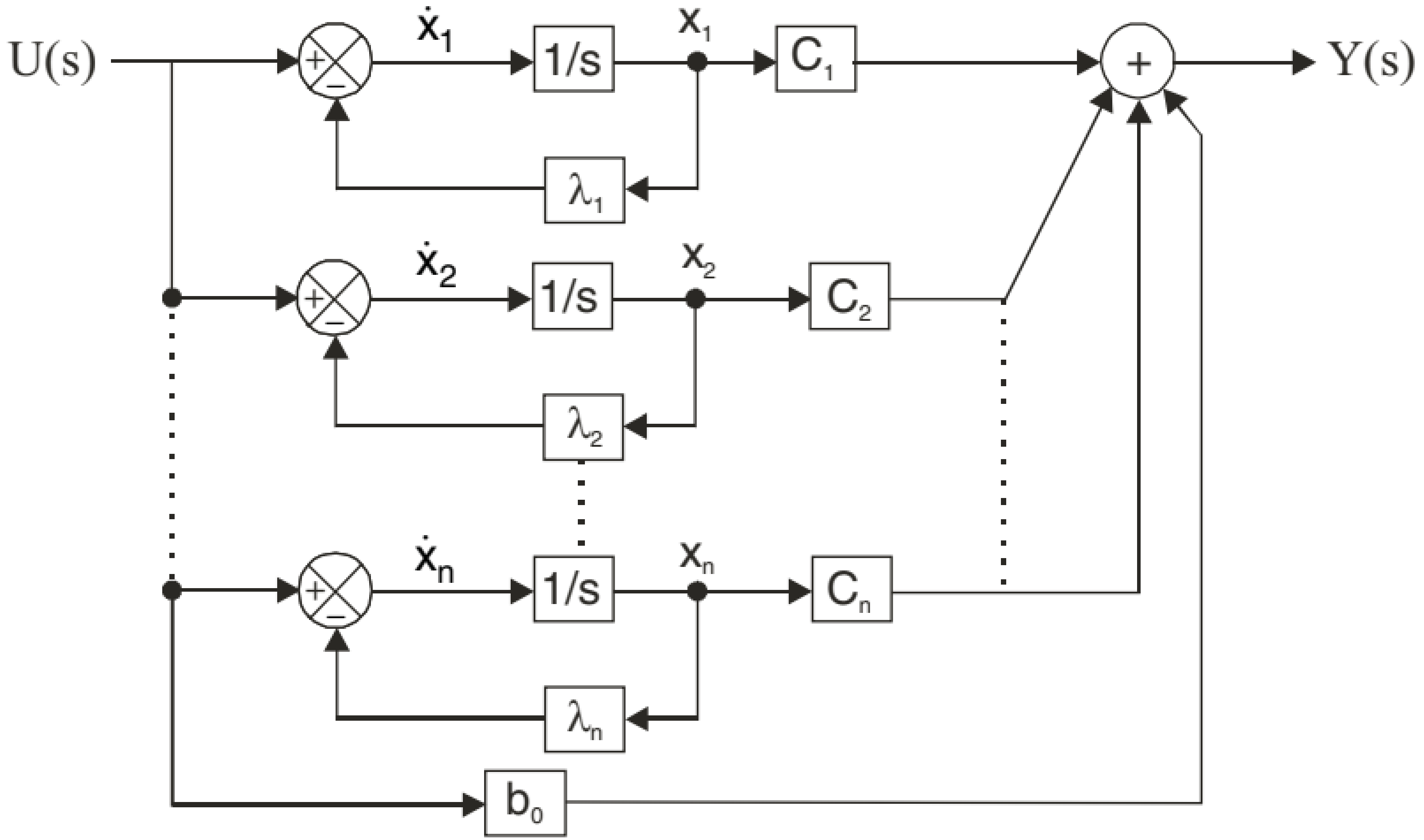
$$= b_0 + \frac{C_1/s}{1 + \lambda_1/s} + \frac{C_2/s}{1 + \lambda_2/s} + \ldots\ldots + \frac{C_n/s}{1 + \lambda_n/s}$$

$$\therefore\ Y(s) = b_0 U(s) + \left[\frac{1/s}{1 + (1/s) \times \lambda_1} \times C_1\right] U(s) + \left[\frac{1/s}{1 + (1/s) \times \lambda_2} \times C_2\right] U(s)$$

$$+ \ldots\ldots + \left[\frac{1/s}{1 + (1/s) \times \lambda_n} \times C_n\right] U(s)$$

State-Space Representation - Diagonal Canonical Form

The equation above can be represented by a block diagram as



State-Space Representation - Diagonal Canonical Form

- Assign state variables at the output of the integrator. The input of the integrator will be the first derivative of the state variable.
- The state equations are formed by adding the incoming signals to the integrator and equating to the first derivative of the state variable.
- The state equations are,

$$\dot{x}_1 = -\lambda_1 x_1 + u$$
$$\dot{x}_2 = -\lambda_2 x_2 + u$$
$$\vdots$$
$$\dot{x}_n = -\lambda_n x_n + u$$

The output equation is, $y = C_1 x_1 + C_2 x_2 + \ldots\ldots\ldots C_n x_n + b_0 u$

The canonical form of state model in the matrix form is given below

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [u]$$

$$y = [C_1 C_2 C_3 \ldots\ldots C_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + [b_0][u]$$

- The advantage of canonical form is that the state equations are independent of each other.
- The disadvantage is that the canonical variables are not physical variables and so they are not available for measurement and control.

State-Space Representation - Jordan Canonical Form

- When a pole of the transfer function has multiplicity, the canonical state model will be in a special form called Jordan canonical form.
- In this form the system matrix A will have a Jordan block of size qxq, corresponding to a pole of value λi with multiplicity q.
- In the Jordan block, the diagonal element will be the poles, and the element just above the diagonal is one.

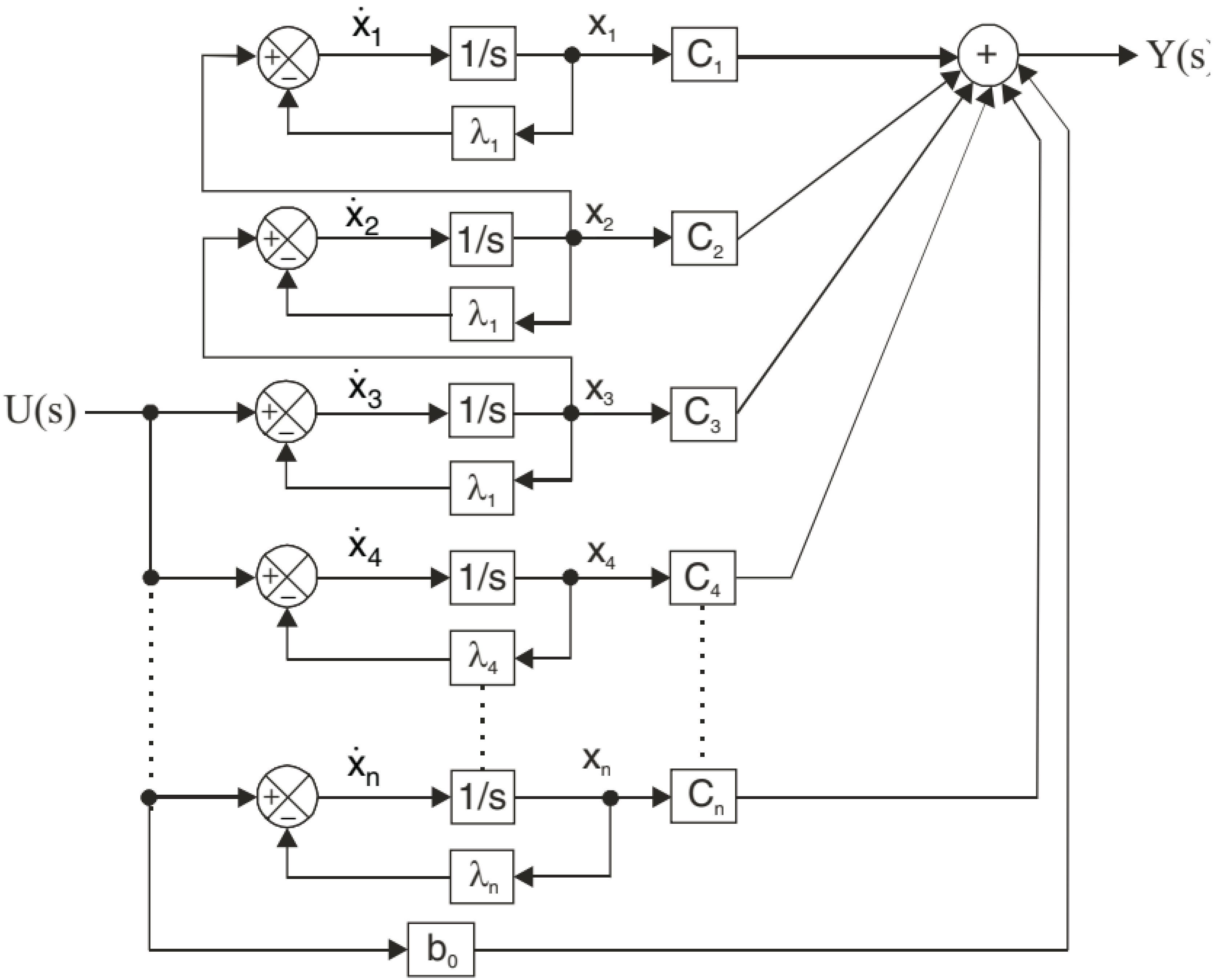
State-Space Representation - Jordan Canonical Form

Consider a system with poles $\lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_5, \ldots.. \lambda_n$, where λ_1 has multiplicity of three.

- The transfer function of the system for this case is given by equation

$$\frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{(s+\lambda_1)^3} + \frac{C_2}{(s+\lambda_1)^2} + \frac{C_3}{s+\lambda_1} + \frac{C_4}{s+\lambda_4} + \ldots\ldots.. + \frac{C_n}{s+\lambda_n}$$

- The block diagram is shown as



$\dot{x}_1$
1/s
x_1
C_1
+
Y(s)
λ_1
$\dot{x}_2$
1/s
x_2
C_2
λ_1
$\dot{x}_3$
1/s
x_3
C_3
U(s)
λ_1
$\dot{x}_4$
1/s
x_4
C_4
λ_4
$\dot{x}_n$
1/s
x_n
C_n
λ_n
b_0

State-Space Representation - Jordan Canonical Form

The input matrix (**B**) and system matrix for this case will be as shown in equation. The system matrix is also denoted as **J**.

Jordan block of size 3×3

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad ; \quad \mathbf{A} = \mathbf{J} = \begin{bmatrix} -\lambda_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\lambda_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_n \end{bmatrix}$$

State-Space Representation - Jordan Canonical Form

State model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\lambda_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$\mathbf{y} = [\mathbf{C_1 C_2 C_3 \ldots\ldots C_n}] \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \\ \vdots \\ \mathbf{x_n} \end{bmatrix} + [\mathbf{b_0}][\mathbf{u}]$$

Transformation of State Model to Canonical Form

Eigenvalues and Eigenvectors

- A nonzero column vector **X** is an eigenvector of a square matrix **A**, if there exists a scalar λ such that

$$\mathbf{AX} = \lambda\,\mathbf{X},$$

  then λ is an eigenvalue (or characteristic value) of **A**.
- Eigenvalue may be zero but the corresponding vector may not be a zero vector.
- The characteristic equation of $n \times n$ matrix A is the nth degree polynomial of equation,

$$|\lambda\,\mathbf{I} - \mathbf{A}| = 0,$$

  where I is the unit matrix.
- Solving the characteristic equation for λ gives the eigenvalues of A.
- The eigenvalues may be real, complex, or multiples of each other.
- Once an eigenvalue is determined, it may be substituted into $\mathbf{AX} = \lambda\,\mathbf{X}$ and then that equation may be solved for the corresponding eigenvector.

Eigenvalues and Eigenvectors

DETERMINATION OF EIGENVECTORS

Case i : Distinct eigenvalues

If the eigenvalues of **A** are all distinct, then we have only one independent eigenvector corresponding to any particular eigenvalue λ_i. The eigenvector corresponding to λ_i may be obtained by taking cofactors of matrix $[\lambda_i \mathbf{I} - \mathbf{A}]$ along any row.

Let, $\mathbf{m}_i$ = Eigenvector corresponding to λ_i

Now the eigenvector $\mathbf{m}_i$ is given by

$$\mathbf{m_i} = \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kn} \end{bmatrix} \; ; \; k = 1 \text{ or } 2 \text{ or }, \ldots n$$

where $C_{k1}, C_{k2}, \ldots C_{kn}$ are cofactors of matrix $[\lambda_i \mathbf{I} - \mathbf{A}]$ along k^{th} row.

Case ii : Multiple eigenvalues

In this case the eigenvectors corresponding to the distinct eigenvalues are evaluated as mentioned in case (i).

If the matrix has repeated eigenvalues with multiplicity "q", then there exists only one independent eigenvector corresponding to that repeated eigenvalue. If λ_i is a repeated eigenvalue, then the independent vector corresponding to λ_i can be evaluated by taking the cofactor of matrix $[\lambda_i \mathbf{I} - \mathbf{A}]$ along any row as mentioned in case (1). The remaining (q–1) eigenvectors can be obtained as shown in equation

Let, $\mathbf{m_p}$ = p^{th} eigenvector corresponding to repeated eigenvalue λ_1.

$$\mathbf{m_p} = \begin{bmatrix} \frac{1}{p!}\frac{d^p}{d\lambda_i^p} C_{k1} \\ \frac{1}{p!}\frac{d^p}{d\lambda_i^p} C_{k2} \\ \vdots \\ \frac{1}{p!}\frac{d^p}{d\lambda_i^p} C_{kn} \end{bmatrix} \; ; \; p = 1, 2, 3, \ldots (q-1)$$

where C_{k1}, C_{k2}, C_{k3} .....C_{kn} are cofactors of matrix $[\lambda_i \mathbf{I} - \mathbf{A}]$ along k^{th} row.

Similarity Transformation

- The square matrices **A** and **B** are said to be similar if a non-singular matrix P exists such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B}$$

- The process of transformation is called similarity transformation and it is a linear transformation.
- The matrix **P** is called transformation matrix.
- Also the matrix, **A** can be obtained from **B** by a similarity transformation with a transformation matrix $\mathbf{P}^{-1}$

$$\text{i.e., } \mathbf{A} = \mathbf{P}\,\mathbf{B}\,\mathbf{P}^{-1}$$

- The similarity transformation can be used for diagonalization of a square matrix.
- If an n × n matrix has n linearly independent eigenvectors (i.e., with distinct eigenvalues) then it can be diagonalized by a similarity transformation.
- If a matrix has multiple eigenvalues then it will not have a complete set of n linearly independent eigenvectors and so it cannot be diagonalized. However such a matrix can be transformed into a Jordan matrix (Jordan canonical form).

Similarity Transformation

- The transformation matrix for diagonalization or converting to Jordan form can be obtained from eigenvectors.
- For a system with n state variables we can find n numbers of eigenvectors $\mathbf{m}_1, \mathbf{m}_{2,} \ldots\ldots \mathbf{m}_n$
- The eigenvectors are column vectors of order (n×1).
- The transformation matrix is obtained by arranging the eigenvectors column-wise.
- This transformation matrix is also called Modal matrix and denoted by M.

Modal matrix, $\mathbf{M} = [\ \mathbf{m}_1\ \ \mathbf{m}_2\ \ \mathbf{m}_3 \ \ldots\ldots\ldots\ldots\ \mathbf{m}_n\ ]$

Transformation of State Model –Canonical Form (Diagonal or Jordan)

Consider the state equation of a system, $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$, where $\mathbf{X}$ is the state variable vector of order n × 1. Let us define a new state variable vector $\mathbf{Z}$, such that $\mathbf{X} = \mathbf{MZ}$, where $\mathbf{M}$ is the Modal matrix or Diagonalization matrix.

The state model of the n^{th} order system is given by

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$$

$$\mathbf{Y} = \mathbf{CX} + \mathbf{DU}$$

On substituting $\mathbf{X} = \mathbf{MZ}$ in the state model of the system, we get

$$\dot{\mathbf{X}} = \mathbf{AMZ} + \mathbf{BU} \qquad \textbf{(1)}$$

$$\mathbf{Y} = \mathbf{CMZ} + \mathbf{DU}$$

Premultiply equation **(1)** by $\mathbf{M}^{-1}$

$$\therefore\ \mathbf{M}^{-1}\dot{\mathbf{X}} = \mathbf{M}^{-1}\mathbf{AMZ} + \mathbf{M}^{-1}\mathbf{BU} \qquad \textbf{(2)}$$

Transformation of State Model –Canonical Form (Diagonal or Jordan)

The relation governing **X** and **Z** is, $\mathbf{X} = \mathbf{MZ}$.

On differentiating equation $\dot{\mathbf{X}} = \mathbf{M}\dot{\mathbf{Z}}$. **(3)**

On premultiplying the equation **(3)** by $\mathbf{M}^{-1}$ we get

$$\mathbf{M}^{-1}\dot{\mathbf{X}} = \dot{\mathbf{Z}} \qquad \textbf{(4)}$$

From equations (2) and (4)

$$\dot{\mathbf{Z}} = \mathbf{M}^{-1}\mathbf{AMZ} + \mathbf{M}^{-1}\mathbf{BU}$$

Let, $\mathbf{M}^{-1}\mathbf{AM} = \boldsymbol{\Lambda}$ (called grammian matrix)

$$\mathbf{M}^{-1}\mathbf{B} = \tilde{\mathbf{B}}$$

$$\mathbf{CM} = \tilde{\mathbf{C}}$$

The Transformed state model

$$\dot{\mathbf{Z}} = \Lambda\mathbf{Z} + \tilde{\mathbf{B}}\mathbf{U}$$

$$\mathbf{Y} = \tilde{\mathbf{C}}\mathbf{Z} + \mathbf{DU}$$

TRANSFER FUNCTION FROM STATE SPACE

Transfer Function from State-Space

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

take the Laplace transform assuming zero initial conditions:

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad (1)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \quad (2)$$

Solving for $\mathbf{X}(s)$ in Eq. (1)

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

Transfer Function from State-Space

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (3)$$

where $\mathbf{I}$ is the identity matrix.

Substituting Eq. (3) into Eq. (2) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

where $\left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]$ is the transfer function matrix, since it relates output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$.

For SISO System $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars. Thus, the transfer function,

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Transfer Function from State-Space

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

Where

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

$$\frac{Y(s)}{U(s)} = \mathbf{C}\,\frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\,\mathbf{B} + \mathbf{D}$$

$$\frac{Y(s)}{U(s)} = \frac{\mathbf{C}\,\operatorname{adj}(s\mathbf{I} - \mathbf{A})\,\mathbf{B} + \mathbf{D}\,\det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

Where the denonimator of the Transfer function

$\det(s\mathbf{I} - \mathbf{A})$ which is the Characteristics equation.

SOLUTION TO TIME INVARIANT AUTONOMOUS SYSTEMS AND FORCED SYSTEMS

Solution for Homogeneous State Equation (Autonomous System)

- Laplace Transform Approach

The homogeneous state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

Taking the Laplace transform of both sides of Equation

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

where $\mathbf{X}(s) = \mathscr{L}[\mathbf{x}]$. Hence,

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0)$$

Premultiplying both sides of this last equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

The inverse Laplace transform of $\mathbf{X}(s)$ gives the solution $\mathbf{x}(t)$. Thus,

$$\boxed{\mathbf{x}(t) = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0)}$$

Solution for Homogeneous State Equation (Autonomous System)

From solution of state equation, $\mathbf{X}(t) = e^{At}\,\mathbf{X}(0)$ we get,

$$\mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = e^{\mathbf{A}t}$$

$$\mathcal{L}[e^{At}] = (s\mathbf{I} - \mathbf{A})^{-1}$$

We know that, $e^{At} = \phi(t)$,

$$\therefore\ \mathcal{L}[e^{At}] = \mathcal{L}[\phi(t)] = \phi(s)$$

where, $\phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ and it is called resolvant matrix.

The solution of state equation is given by,

$$\mathbf{X}(t) = e^{At}\,\mathbf{X}(0)$$

$$\therefore\ \mathbf{X}(t) = \mathcal{L}^{-1}[\phi(s)]\ \mathbf{X}(0)$$

where, $\phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$

Solution for Non Homogeneous State Equation (Forced System)

Laplace Transform Approach

The solution of the nonhomogeneous state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

can also be obtained by the Laplace transform approach. The Laplace transform of this last equation yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Premultiplying both sides of this last equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \phi(s)\,\mathbf{X}(0) + \phi(s)\,\mathbf{B}\,\mathrm{U}(s) \qquad \text{where, } \phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

Solution for Non Homogeneous State Equation (Forced System)

On taking inverse Laplace transform of equation, solution is

$$\mathbf{X}(t) = \phi(t)\,\mathbf{X}(0) + \mathcal{L}^{-1}\left[\phi(s)\,\mathbf{B}\,\mathbf{U}(s)\right]$$

OR

Using the relationship given by Equation $\mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = e^{\mathbf{A}t}$

$$\mathbf{X}(s) = \mathscr{L}\left[e^{\mathbf{A}t}\right]\mathbf{x}(0) + \mathscr{L}\left[e^{\mathbf{A}t}\right]\mathbf{B}\mathbf{U}(s)$$

The inverse Laplace transform of this last equation can be obtained by use of the convolution integral as follows:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Solution in Terms of $\mathbf{x}(t_0)$. Thus far we have assumed the initial time to be zero. If, however, the initial time is given by t_0 instead of 0, then the solution to Equation must be modified to

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Computation of State Transition Matrix Using Laplace Transform and Cayley-Hamilton Method

State Transition Matrix - Laplace Transform Method

State Transition Matrix, $e^{At} = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \phi(t),$

$$\text{Let } \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$s\mathbf{I}-\mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$|s\mathbf{I}-\mathbf{A}| = s^2+3s+2$$

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

State Transition Matrix - Laplace Transform Method

$$[s\mathbf{I}-\mathbf{A}]^{-1}=\begin{bmatrix}\dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)}\\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{2}{(s+1)}+\dfrac{-1}{(s+2)} & \dfrac{1}{(s+1)}+\dfrac{-1}{(s+2)}\\ \dfrac{-2}{(s+1)}+\dfrac{2}{(s+2)} & \dfrac{-1}{(s+1)}+\dfrac{2}{(s+2)}\end{bmatrix}$$

$$e^{\mathbf{A}t}=\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

State Transition Matrix - Cayley-Hamilton Method

The CAYLEY HAMILTON theorem says that "every square matrix satisfies its own characteristic equation".

Characteristic equation is given by $|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_1 \lambda^{n-1} + \ldots\ldots\ldots + a_{n-1} \lambda + a_n = 0$

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \cdots .. + a_{n-1}\mathbf{A} + a_n \mathbf{I} = 0$$

- Consider a square matrix A= $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Then the characteristic equation is given by $\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right|$=0

$=\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix}$=0 ie. $\lambda(\lambda+3)+2 = \lambda^2 + 3\lambda + 2 = 0$

Same way, $\mathbf{A}^2 + 3\mathbf{A} + 2$ also will be equal to 0

State Transition Matrix - Cayley-Hamilton Method

If **A** is a non-singular n x n matrix, then the corresponding polynomial is

$$f(\mathbf{A}) = \alpha_o\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \cdots + \alpha_{n-1}\mathbf{A}^{n-1} = \boldsymbol{e}^{\mathbf{A}\boldsymbol{t}}$$

We can determine $\alpha_0, \alpha_1, \quad \dots \dots$ and α_{n-1} by writing

$f(\lambda_i)$= α_0+$\alpha_1\lambda_i$+$\alpha_2\lambda_i^2$+……..+$\alpha_{n-1}\lambda_i^{n-1}$ (n simultaneous equations)

for *i*=1,2,…..up to n, and solving them.

State Transition Matrix - Cayley-Hamilton Method

Procedure to find matrix polynomial $f(\mathbf{A})$ *where* $f(\mathbf{A}) = \boldsymbol{\phi}(\boldsymbol{t})$

1. Find the eigenvalues of system matrix A

2. If all eigenvalues are different, solve n simultaneous equations

   $f(\lambda_i) = \alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 + \ldots\ldots.. + \alpha_{n-1}\lambda_i^{n-1}$ to find coefficients of $\alpha_0, \alpha_1,$ … … . and α_{n-1}

3. Substitute the coefficients $\alpha_0, \alpha_1,$ … … . *and* α_{n-1} to obtain the polynomial $f(A) = \alpha_0 \mathrm{I} + \alpha_1 A + \alpha_2 A^2 + \ldots \ldots \ldots + \alpha_{n-1}A^{n-1}$

State Transition Matrix - Cayley-Hamilton Method

Q1:- Obtain the STM for the state model whose A matrix is given by

A= $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: **A** is a non singular matrix

Step 1: Find eigen values

$|(\lambda I - \mathbf{A})|$=0 ie $\left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right| = \left|\begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix}\right| = (\lambda(\lambda + 3)+2$

$$\lambda^2 + 3\lambda + 2 = 0$$

So eigen values are $\lambda_1 = -1, \ \lambda_2 = -2$

Step 2: $f(\lambda_i)= \alpha_0+\alpha_1\lambda_i+\alpha_2\lambda_i^2+$……..$+\alpha_{n-1}\lambda_i^{n-1}$

Here n=2; $f(\lambda_i)= \alpha_0+\alpha_1\lambda_i = e^{\lambda_i t}$;

$f(-1)= \alpha_0-\alpha_1 = e^{-t}$ ------------------------------------(1)

and $f(-2)= \alpha_0-2\alpha_1= e^{-2t}$ ………………..(2)

(1)-(2)

$\boldsymbol{e^{-t}}$- $\boldsymbol{e^{-2t}}$$=(\alpha_0-\alpha_1)$ - $(\alpha_0-2\alpha_1)$= $\boldsymbol{\alpha_1}$-----------------(3)

So, $\alpha_0-(e^{-t}$- $e^{-2t}) = e^{-t}$

$$\boldsymbol{\alpha_0}= (e^{-t} - e^{-2t}) + e^{-t} = \boldsymbol{2e^{-t} - e^{-2t}}$$

$\alpha_1 = e^{-t}$- e^{-2t} and α_0= $2e^{-t}$- e^{-2t}

Step 3: Substitute the coefficients $\alpha_0, and\ \alpha_1$, to obtain the polynomial

$$f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$$

$$=2e^{-t} - e^{-2t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} - e^{-2t} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

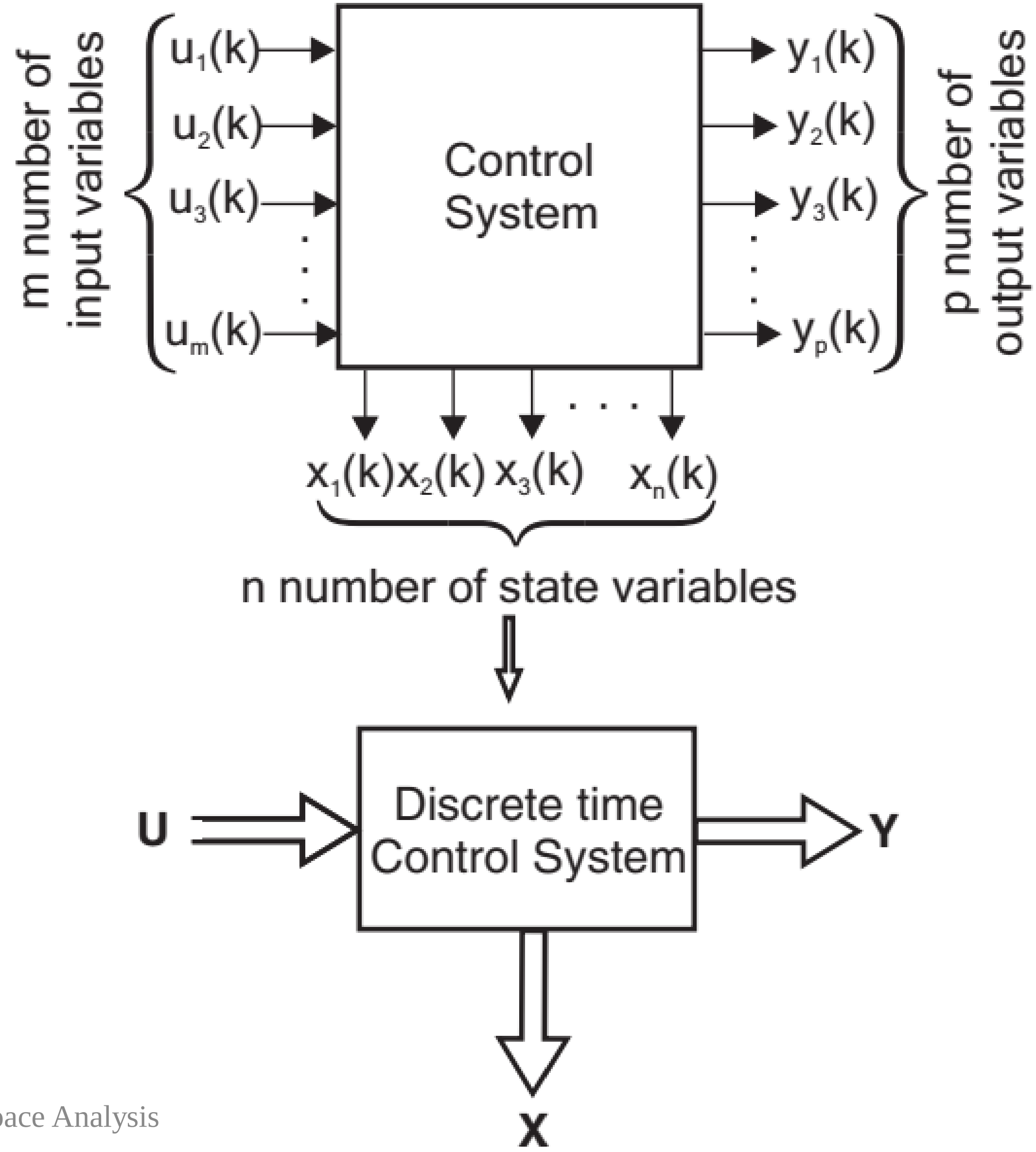
$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & 0 \\ 0 & 2e^{-t} - e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 & e^{-t} - e^{-2t} \\ -2e^{-t} + 2\,e^{-2t} & -3e^{-t} + 3\,e^{-2t} \end{bmatrix}$$

$$\boldsymbol{f}(\mathbf{A}) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2\,e^{-2t} & -e^{-t} + 2\,e^{-2t} \end{bmatrix}$$

STATE SPACE REPRESENTATION OF DISCRETE TIME SYSTEMS

State Space Representation of Discrete Time Systems

- The state variable analysis techniques of continuous time systems can be extended to the discrete time system.
- The discrete form of state space representation is quite analogous to the continuous form.
- In the state variable formulation of a discrete time system, in general, a system consists of m-inputs, p-outputs and n-state variables.
- The state space representation of discrete-time system may be visualized as shown in figure.



State Space Representation of Discrete Time Systems

Let, State variables = $x_1(k), x_2(k), x_3(k), \ldots\ldots\ldots\; x_n(k)$

Input variables = $u_1(k), u_2(k), u_3(k), \ldots\ldots\ldots\; u_m(k)$

Output variables = $y_1(k), y_2(k), y_3(k), \ldots\ldots\ldots\; y_p(k)$

The different variables may be represented by the vectors (column matrix) as shown below.

Input vector $U(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_m(k) \end{bmatrix}$; Output vector $\mathbf{Y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_p(k) \end{bmatrix}$; State variable vector $\mathbf{X}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$

> ***Note :*** *The simplified notation x(k), y(k) and u(k) are used to denote x(kT), y(kT) and u(kT) respectively. Also for convenience the variables are denoted* $x_1, x_2, x_3, \ldots ; y_1, y_2, y_3, \ldots$ *and* $u_1, u_2, u_3 \ldots$

The state equation of a discrete time system is a set of n-numbers of first order difference equations.

$$x_1(k+1) = f_1(x_1, x_2, \ldots\ldots x_n \;;\; u_1, u_2, \ldots\ldots u_m)$$

$$x_2(k+1) = f_2(x_1, x_2, \ldots\ldots x_n \;;\; u_1, u_2, \ldots\ldots u_m)$$

$$\vdots$$

$$x_n(k+1) = f_n(x_1, x_2, \ldots\ldots x_n \;;\; u_1, u_2, \ldots\ldots u_m)$$

State Space Representation of LTI - Discrete Time Systems

For linear time invariant discrete time systems the above difference equations can be expressed as a linear combination of state variables and inputs.

$$x_1(k+1) = a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \ldots\ldots + b_{1m}u_m$$

$$x_2(k+1) = a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \ldots\ldots + b_{2m}u_m$$

$$\vdots$$

$$x_n(k+1) = a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \ldots\ldots + b_{nm}u_m$$

where, the coefficients a_{ij} and b_{ij} are constants.

In the matrix form the above equations can be expressed as,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots\cdots & b_{1m} \\ b_{21} & b_{22} & \cdots\cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots\cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

State Space Representation of LTI - Discrete Time Systems

- The matrix equation can be written in the vector notation as

$$\mathbf{X}(k+1) = \mathbf{A}\mathbf{X}(k) + \mathbf{B}\mathbf{U}(k)$$

where,
$\mathbf{X}(k)$ = State vector of order $(n \times 1)$
$\mathbf{U}(k)$ = Input vector of order $(m \times 1)$
$\mathbf{A}$ = System matrix of order $(n \times n)$
$\mathbf{B}$ = Input matrix of order $(n \times m)$

- This equation is the state equation of (linear time-invariant) discrete-time system.
- The output at any discrete time instant, k are functions of state variables and inputs.
- Hence the output variables of the linear time-invariant system can be expressed as a linear combination of state variables and inputs.

$$y_1 = c_{11}x_1 + c_{12}x_2 + \ldots\ldots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \ldots\ldots + d_{1m}u_m$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + \ldots\ldots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \ldots\ldots + d_{2m}u_m$$
$$\vdots$$
$$y_p = c_{p1}x_1 + c_{p2}x_2 + \ldots\ldots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \ldots\ldots + d_{pm}u_m$$

where, the coefficients c_{ij} and d_{ij} are constants.

In the matrix form the above equations can be expressed as,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots\cdots & c_{1n} \\ c_{21} & c_{22} & \cdots\cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots\cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots\cdots & d_{1m} \\ d_{21} & d_{22} & \cdots\cdots & d_{2m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots\cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

The matrix equation can be written in the vector notation as,

$$\mathbf{Y}(k) = \mathbf{CX}(k) + \mathbf{DU}(k)$$

where, $\mathbf{X}(k)$ = State vector of order $(n \times 1)$

$\mathbf{U}(k)$ = Input vector of order $(m \times 1)$

$\mathbf{Y}(k)$ = Output vector of order $(p \times 1)$

$\mathbf{C}$ = Output matrix of order $(p \times n)$

$\mathbf{D}$ = Transmission matrix of order $(p \times m)$

The equation is the output equation of (linear time invariant) discrete time system.

State Space Representation of LTI - Discrete Time Systems

- The state equation and the output equation are together called as the state model of the system.
- Hence the state model of the discrete-time system is given by the following equations

$$\mathbf{X}(k+1) = \mathbf{A}\,\mathbf{X}(k) + \mathbf{B}\,\mathbf{U}(k) \quad \text{........ State equation}$$

$$\mathbf{Y}(k) = \mathbf{C}\,\mathbf{X}(k) + \mathbf{D}\,\mathbf{U}(k) \quad \text{......... Output equation}$$

State Space Representation of Discrete Time Systems- Problem

Example A discrete time system is described by the difference equation

$$y(k+3) + 3y(k+2) + 2y(k+1) + y(k) = 5u(k)$$

Define the state variables as:

$$x_1(k) = y(k)$$
$$x_2(k) = y(k+1)$$
$$x_3(k) = y(k+2)$$

we have,

$$x_1(k+1) = y(k+1) = x_2(k)$$
$$x_2(k+1) = y(k+2) = x_3(k)$$
$$x_3(k+1) = y(k+3)$$
$$x_3(k+1) + 3x_3(k) + 2x_2(k) + x_1(k) = 5u(k)$$

State Space Representation of Discrete Time Systems

The state equations are:

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = x_3(k)$$

$$x_3(k+1) = -x_1(k) - 2x_2(k) - 3x_3(k) + 5u(k)$$

Output equation

$$y(k) = x_1(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0 \quad 0]\mathbf{x}(k) + [0]u(k)$$

Transfer Function from Difference Equation

Let the difference equation governing the discrete system be,

$$y(k+3) + 3y(k+2) + 2y(k+1) + y(k) = 5u(k)$$

Taking z-transform

$$z^3Y(z) + 3z^2Y(z) + 2zY(z) + Y(z) = 5U(z)$$

$$(z^3 + 3z^2 + 2z + 1)\,Y(z) = 5U(z)$$

$$\frac{Y(z)}{U(z)} = \frac{5}{z^3 + 3z^2 + 2z + 1}$$

State Space Representation of Discrete Time Systems

Q: A discrete time system has the transfer function

$$\frac{Y(z)}{U(z)} = \frac{4z^3 - 12z^2 + 13z - 7}{(z-1)^2(z-2)}$$

Determine the state model of the system in 2 canonical models.

State Space Representation of Discrete Time Systems
Phase Variable Form or Controllable Canonical Form

Solution: Let $\frac{Y(z)}{U(z)} = \frac{4z^3-12z^2+13z-7}{(z-1)^2(z-2)} = \frac{Y(z)}{X(z)}\frac{X(z)}{U(z)}$

$$\frac{Y(z)}{X(z)} = \frac{4z^3-12z^2+13z-7}{1}$$

and $\frac{X(z)}{U(z)} = \frac{1}{(z-1)^2(z-2)} = \frac{1}{z^3-4z^2+5z-2}$

$\frac{X(z)}{U(z)} = \frac{1}{z^3-4z^2+5z-2}$ ie $U(z) = (z^3 - 4z^2 + 5z - 2)X(z)$

$$U(z) = \left(z^3X(z) - 4z^2X(z) + 5zX(z) - 2X(z)\right)$$

Converting to time domain from z domain; Taking inverse z-transform

$$Z[f(t - mT)] = z^{-m}F(z)$$

$$u(k) = x(k + 3) - 4x(k + 2) + 5x(k + 1) - 2x(k)$$

Let $x(k) = x_1(k)$; $x(k+1) = x_2(k)$; and $x(k+2) = x_3(k)$

From that $x(k + 1) = x_1(k + 1) = x_2(k)$

$$x(k + 2) = x_2(k + 1) = x_3(k)$$

and $x(k + 3) = x_3(k + 1) = 2x_1(k) - 5x_2(k) + 4x_3(k) + u(k)$

State Equation

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_3(k)$$
$$x_3(k+1) = 2x_1(k) - 5x_2(k) + 4x_3(k) + u(k)$$

State Equation in matrix form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Now,

$$\frac{Y(z)}{X(z)} = \frac{4z^3 - 12z^2 + 13z - 7}{1}$$

$$\mathrm{Y(z)} = [4z^3 - 12z^2 + 13z - 7]\,\mathrm{X(z)}$$

Taking inverse Z-Transform

$\mathrm{y(k)} = 4\mathrm{x}(k+3) - 12\mathrm{x}(k+2) + 13\mathrm{x}(k+1) - 7\mathrm{x}(k)$ in discrete time

$$\mathrm{y(k)} = 4\,(2x_1(k) - 5x_2(k) + 4x_3(k) + u(k)) - 12x_3(k) + 13\,x_2(k) - 7x_1(k)$$

$$\mathrm{y(k)} = 8x_1 - 7x_1 - 20x_2 + 13x_2 + 16x_3 - 12x_3 + 4u$$

$$\mathrm{Y(k)} = x_1 - 7x_2 + 4x_3 + 4u$$

$\mathrm{y(k)} = [1 \quad -7 \quad 4] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} u(k)$ Output equation

Solution of Discrete Time State Equation

The state equation of discrete time system is given by

$$\mathbf{X}(k+1) = \mathbf{A}\,\mathbf{X}(k) + \mathbf{B}\,\mathbf{U}(k)$$

When k = 0, the equation can be written as

$$\mathbf{X}(1) = \mathbf{A}\,\mathbf{X}(0) + \mathbf{B}\,\mathbf{U}(0)$$

When k = 1, the equation can be written as

$$\mathbf{X}(2) = \mathbf{A}\,\mathbf{X}(1) + \mathbf{B}\,\mathbf{U}(1)$$

On substituting for **X**(1) from equation

$$\mathbf{X}(2) = \mathbf{A}\,[\mathbf{A}\,\mathbf{X}(0) + \mathbf{B}\,\mathbf{U}(0)] + \mathbf{B}\mathbf{U}(1)$$

$$\therefore \mathbf{X}(2) = \mathbf{A}^2\,\mathbf{X}(0) + \mathbf{A}\,\mathbf{B}\,\mathbf{U}(0) + \mathbf{B}\mathbf{U}(1)$$

On continuing this analysis till **X**(k) we get,

$$\mathbf{X}(k) = \mathbf{A}^k\,\mathbf{X}(0) + \mathbf{A}^{(k-1)}\,\mathbf{B}\,\mathbf{U}(0) + \mathbf{A}^{(k-2)}\,\mathbf{B}\mathbf{U}(1) + \ldots\ldots$$

$$\ldots\ldots + \mathbf{A}\mathbf{B}\mathbf{U}(k-2) + \mathbf{B}\mathbf{U}(k-1)$$

Solution of Discrete Time State Equation

The matrix equation is the solution of discrete time state equation. The matrix $\mathbf{A}^k$ is called the state transition matrix of discrete time system and it is also denoted by $\phi(k)$.

On substituting, $\mathbf{A}^k = \phi(k)$ and $\phi(0) = \mathbf{I}$ in equation

$$\mathbf{X}(k) = \phi(k)\,\mathbf{X}(0) + \phi(k-1)\,\mathbf{B}\,\mathbf{U}(0) + \phi(k-2)\,\mathbf{BU}(1) + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots + \phi(1)\,\mathbf{BU}(k-2) + \phi(0)\,\mathbf{BU}(k-1)$$

$$\therefore\ \mathbf{X}(k) = \phi(k)\,\mathbf{X}(0) + \sum_{j=0}^{k-1} \phi(k-1-j)\,\mathbf{BU}(j)$$

SOLUTION OF DISCRETE TIME STATE EQUATION USING Z-TRANSFORM

Consider the state equation of the discrete time system

$$\mathbf{X}(k+1) = \mathbf{A}\,\mathbf{X}(k) + \mathbf{B}\,\mathbf{U}(k)$$

On taking $\mathcal{Z}$-transform of equation

$$z\,\mathbf{X}(z) - z\,\mathbf{X}(0) = \mathbf{A}\,\mathbf{X}(z) + \mathbf{B}\,\mathbf{U}(z)$$

$$z\,\mathbf{X}(z) - \mathbf{A}\,\mathbf{X}(z) = z\,\mathbf{X}(0) + \mathbf{B}\,\mathbf{U}(z)$$

$$(z\mathbf{I} - \mathbf{A})\,\mathbf{X}(z) = z\,\mathbf{X}(0) + \mathbf{B}\,\mathbf{U}(z)$$

On premultiplying the equation by $(z\mathbf{I}-\mathbf{A})^{-1}$ we get

$$\mathbf{X}(\mathbf{z}) = (z\mathbf{I} - \mathbf{A})^{-1}\; z\,\mathbf{X}(0) + (z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(z)$$

On taking inverse $\mathcal{Z}$-transform of equation we get $\mathbf{X}(k)$

$$\therefore\; \mathbf{X}(k) = \mathcal{Z}^{-1}\{\mathbf{X}(z)\} = \mathcal{Z}^{-1}\{(z\mathbf{I} - \mathbf{A})^{-1} z\mathbf{X}(0) + (z\mathbf{I} - \mathrm{A})^{-1}\,\mathbf{BU}(z)\}$$

$$\mathrm{X}(k) = \mathcal{Z}^{-1}\{(z\mathbf{I} - \mathbf{A})^{-1} z\}\mathbf{X}(0) + \mathcal{Z}^{-1}\{(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{BU}(z)\}$$

The equation is the solution of discrete time state equation.

PROPERTIES OF STATE TRANSITION MATRIX OF DISCRETE TIME SYSTEM

1. $\phi(0) = \mathbf{I}$
2. $\phi^{-1}(k) = \phi(-k)$
3. $\phi(k, k_0) = \phi(k-k_0) = \mathbf{A}^{(k-k_0)}$; where, $k > k_0$

COMPUTATION OF STATE TRANSITION MATRIX

The state transition matrix $\mathbf{A}^k$ can be computed by any one of the following methods.

Method 1 : Computation of $\mathbf{A}^k$ using $\mathcal{Z}$-transform

Method 2 : Computation of $\mathbf{A}^k$ by canonical transformation

Method 3 : Computation of $\mathbf{A}^k$ by Cayley - Hamilton theorem

The computation of $\mathbf{A}^k$ using $\mathcal{Z}$-transform have been dealt in this section.

Comparing the two solutions

State transition matrix, $A^k = \mathcal{Z}^{-1}\{(z\mathbf{I} - \mathbf{A})^{-1}z\}$

The equation can be used to compute the state transition matrix, $\mathbf{A}^k$.

STABILITY OF SAMPLED DATA SYSTEMS

Stability Analysis of Closed Loop System in Z-plane

An initially relaxed (all the initial conditions of the system are zero) LTI system is said to be BIBO stable if for every bounded input, the output is also bounded.

However, the stability of the following closed loop system

$$\frac{C(z)}{R(Z)} = \frac{G(z)}{1 + GH(z)}$$

can be determined from the location of closed loop poles in z-plane which are the roots of the characteristic equation

$$1 + GH(z) = 0$$

1. For the system to be stable, the closed loop poles or the roots of the characteristic equation must lie within the unit circle in z-plane. Otherwise the system would be unstable.
2. If a simple pole lies at $|z| = 1$, the system becomes marginally stable. Similarly if a pair of complex conjugate poles lie on the $|z| = 1$ circle, the system is marginally stable. Multiple poles on unit circle make the system unstable.

Stability Analysis of closed loop system in z-plane - Jury's Stability Test

- The Jury's stability test is used to determine whether the roots of the characteristic polynomial lie within a unit circle or not.
- The Jury's test consists of two parts.
- One simple test for necessary condition for stability and another test for sufficient condition for stability..

Let F(z) be the n^{th} order characteristic polynomial of a sampled data control system.

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots\ldots + a_2 z^2 + az + a_0 = 0$$

where, $a_n > 0$ and $a_0, a_1, a_2, \ldots\ldots a_n$ are constant coefficients.

The necessary conditions to be satisfied for the stability of the system with characteristic polynomial, F(z) are

$$\mathbf{F(1) > 0 \text{ and } (-1)^n F(-1) > 0}$$

For sufficient conditions, prepare a table as shown below using the coefficients of the characteristic polynomial F(z). The table consists of (2n − 3) rows, where n is the order of the characteristic equation.

Row	z^0	z^1	z^2	......	z^{n-k}	.....	z^{n-2}	z^{n-1}	z^n
1	a_0	a_1	a_2	......	a_{n-k}	.....	a_{n-2}	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	......	a_k	.....	a_2	a_1	a_0
3	b_0	b_1	b_2	......	......	.....	b_{n-2}	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	......	......	.....	b_1	b_0	
5	c_0	c_1	c_2	......	......	.....	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	......	......	.....	c_0		
⋮	⋮	⋮							
2n – 5	s_0	s_1	s_2	s_3					
2n – 4	s_3	s_2	s_1	s_0					
2n – 3	r_0	r_1	r_2						

Stability Analysis of closed loop system in z-plane - Jury's Stability Test

In the above table the elements of row-1 are formed using the coefficients of characteristic polynomial and the row-2 is formed by arranging the elements of row-1 in the reverse order.

The k^{th} element of row-3 is given by, $b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$

$$\therefore b_0 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix} \; ; \quad b_1 = \begin{vmatrix} a_0 & a_{n-1} \\ a_n & a_1 \end{vmatrix} \; ; \quad b_2 = \begin{vmatrix} a_0 & a_{n-2} \\ a_n & a_2 \end{vmatrix} \text{ and so on}$$

The row-4 is formed by arranging the elements of row-3 in reverse order.

The k^{th} element of row-5 is given by, $c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$

$$\therefore c_0 = \begin{vmatrix} b_0 & b_{n-1} \\ b_{n-1} & b_0 \end{vmatrix} \; ; \quad c_1 = \begin{vmatrix} b_0 & b_{n-2} \\ b_{n-1} & b_1 \end{vmatrix} \; ; \quad c_2 = \begin{vmatrix} b_0 & b_{n-3} \\ b_{n-1} & b_2 \end{vmatrix} \text{ and so on}$$

The row-6 is formed by arranging the elements of row-5 in reverse order.

Stability Analysis of closed loop system in z-plane - Jury's Stability Test

The first column elements of the table are used to check the following (n –1) conditions. These (n –1) conditions are the sufficient conditions for stability of the system.

$$\left.\begin{array}{l} |\mathbf{a}_0| < |\mathbf{a}_n| \\ |\mathbf{b}_0| > |\mathbf{b}_{n-1}| \\ |\mathbf{c}_0| > |\mathbf{c}_{n-2}| \\ \quad \vdots \\ |\mathbf{r}_0| > |\mathbf{r}_2| \end{array}\right\} \textbf{(n – 1) conditions}$$

If the necessary and sufficient conditions are satisfied then all the poles of the system lies inside the unit circle in z-plane and so the system is stable.

If even one of the condition is not satisfied then the system is unstable.