NONLINEAR SYSTEMS ANALYSIS

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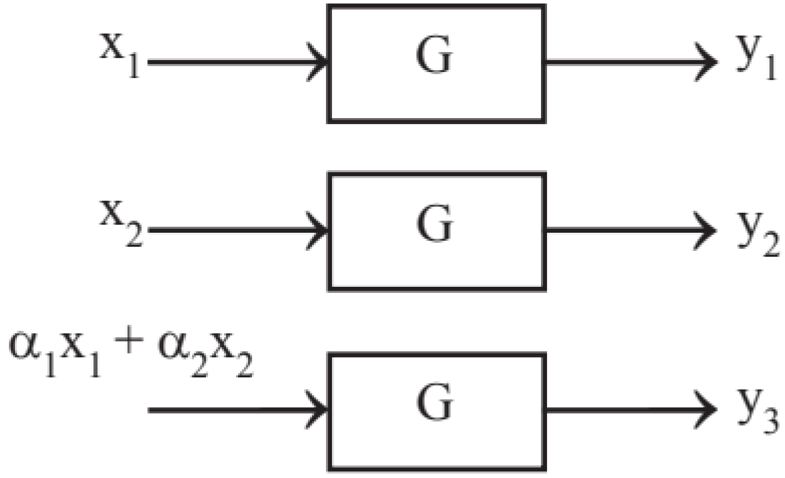
Nonlinear Systems

The nonlinear systems are systems which does not obey the principle of superposition. The linear systems are systems which satisfy the principle of superposition.

The principle of superposition implies that if a system has responses $y_1(t)$ and $y_2(t)$ to any two inputs $x_1(t)$ and $x_2(t)$ respectively then the system response to the linear combination of these inputs $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is given by the linear combination of the individual outputs, i.e, $\alpha_1 y_1(t) + \alpha_2 y_2(t)$, where α_1 and α_2 are constants.



To satisfy the principle of superposition, $y_3 = \alpha_1 y_1 + \alpha_2 y_2$

Nonlinear Systems

EXAMPLE

a) The response of a system is, $y = ax + b\frac{dx}{dt}$, Test whether the system is linear or nonlinear.

b) The response of a system is, $y = ax^2 + e^{bx}$. Test whether the system is linear or nonlinear.

SOLUTION

a) Let x_1 and x_2 be the two inputs to the system and y_1 and y_2 be their responses, respectively.

Given that $y = ax + b\frac{dx}{dt}$

When $x = x_1$, $y = y_1$, $\therefore\ y_1 = ax_1 + b\frac{dx_1}{dt}$

When $x = x_2$, $y = y_2$, $\therefore\ y_2 = ax_2 + b\frac{dx_2}{dt}$

Nonlinear Systems

Consider a linear combination of inputs $\alpha_1 x_1 + \alpha_2 x_2$ and let the response of the system for this linear combination of inputs be y_3.

When $x = \alpha_1 x_1 + \alpha_2 x_2$, $\quad y = y_3$

$$\therefore \; y_3 = a(\alpha_1 x_1 + \alpha_2 x_2) + b\,\frac{d}{dt}(\alpha_1 x_1 + \alpha_2 x_2)$$

$$= \alpha_1 a x_1 + \alpha_2 a x_2 + \alpha_1 b\,\frac{dx_1}{dt} + \alpha_2 b\,\frac{dx_2}{dt}$$

Consider the same linear combination of output, $\alpha_1 y_1 + \alpha_2 y_2$.

$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1\left[a x_1 + b\,\frac{dx_1}{dt}\right] + \alpha_2\left[a x_2 + b\,\frac{dx_2}{dt}\right]$$

$$= \alpha_1 a x_1 + \alpha_2 a x_2 + \alpha_1 b\,\frac{dx_1}{dt} + \alpha_2 b\,\frac{dx_2}{dt}$$

It is observed that $y_3 = \alpha_1 y_1 + \alpha_2 y_2$. Hence the system is linear.

b) Let x_1 and x_2 be two inputs to the system and y_1 and y_2 be their responses respectively.

Given that $y = ax^2 + e^{bx}$

$$\text{When } x = x_1, \quad y = y_1, \quad \therefore \; y_1 = ax_1^2 + e^{bx_1}$$

$$\text{When } x = x_2, \quad y = y_2, \quad \therefore \; y_2 = ax_2^2 + e^{bx_2}$$

Consider a linear combination of inputs $\alpha_1 x_1 + \alpha_2 x_2$ and let the response of the system for this linear combination of inputs be y_3.

$$\text{When} \quad x = \alpha_1 x_1 + \alpha_2 x_2, \; y = y_3$$

$$\therefore \; y_3 = a(\alpha_1 x_1 + \alpha_2 x_2)^2 + e^{b(\alpha_1 x_1 + \alpha_2 x_2)}$$

$$= a(\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 + 2\alpha_1 x_1 \alpha_2 x_2) + e^{\alpha_1 bx_1}.e^{\alpha_2 bx_2}$$

$$= a\alpha_1^2 x_1^2 + a\alpha_2^2 x_2^2 + 2a\alpha_1\alpha_2 x_1 x_2 + e^{\alpha_1 bx_1}.e^{\alpha_2 bx_2}$$

Consider the same linear combination of output , $\alpha_1 y_1 + \alpha_2 y_2$

$$\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1\left[ax_1^2 + e^{bx_1}\right] + \alpha_2\left[ax_2^2 + e^{bx_2}\right]$$

$$= a\alpha_1 x_1^2 + \alpha_1 e^{bx_1} + a\alpha_2 x_2^2 + \alpha_2 e^{bx_2}$$

It is observed that $y_3 \neq \alpha_1 y_1 + \alpha_2 y_2$. Hence the system is nonlinear.

Nonlinear Systems

- In all practical engineering systems, there will be always some nonlinearity due to friction, inertia, stiffness, backlash, hysteresis, saturation and dead-zone.
- The effect of the nonlinear components can be avoided by restricting the operation of the component over a narrow limited range.
- Moreover most of the automatic control systems operate within a narrow range, e.g. the speed controller of an electric drive for constant speed operation of 1500 rpm will be required to operate between 1450 to 1550 rpm.
- Similarly, automatic voltage controller will be operating within ± 5% of the specified voltage.
- Thus the characteristics of components may be considered as linear over this limited range.

Nonlinear Systems

- Further, some components behave linearly over its working range, e.g., a spring when loaded, gets extended.
- As the load is being increased the load-displacement curve is linear within the working range.
- However, when the load is increased beyond the maximum of the working range, the spring material starts to yield and it becomes permanently deformed.
- It can be concluded that the spring behaves linearly over its working range and beyond this range it is nonlinear.

Nonlinear Systems-Characteristics

1. The response of nonlinear system to a particular test signal is no guide to their behaviour to other inputs, since the principle of superposition does not holds good for nonlinear systems.
2. The nonlinear system response may be highly sensitive to input amplitude. The stability study of nonlinear systems requires the information about the type and amplitude of the anticipated inputs, initial conditions, etc., in addition to the usual requirement of the mathematical model.
3. The nonlinear systems may exhibit limit cycles which are self sustained oscillations of fixed frequency and amplitude.
4. The nonlinear systems may have jump resonance in the frequency response.
5. The output of a nonlinear system will have harmonics and sub-harmonics when excited by sinusoidal signals.
6. The nonlinear systems will exhibit phenomena like frequency entrainment and asynchronous quenching.

Nonlinear Systems Characteristics

Super position principle is not valid

- Non linear systems do not obey super position theorem.
- Response due to different magnitudes of initial conditions or different magnitudes of same inputs may not carry similarity from one another.
- Response pattern for different types of inputs may be drastically different.

Multiple equilibrium states and equilibrium zones

- Linear systems have only one equilibrium point, the origin.
- Nonlinear systems may have several equilibrium points and from a given initial state which of the equilibrium points is reached as time evolves depends on several reasons such as the proximity to the equilibrium points, the magnitude of the disturbance etc.
- Some non linear systems may exhibits equilibrium zones rather than equilibrium points.

Nonlinear Systems-Characteristics

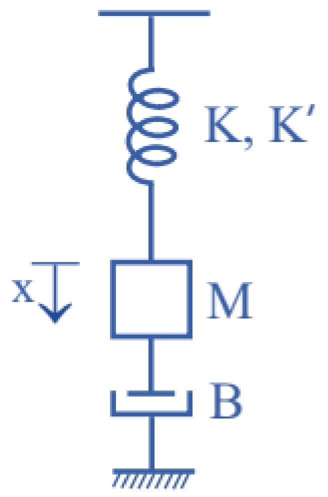
Frequency -Amplitude Dependence

- The frequency-amplitude dependence is one of the most fundamental characteristics of the oscillations of nonlinear systems.
- The frequency amplitude dependence can be best studied by considering the mechanical system shown in figure in which the spring is nonlinear.
- The differential equation governing the dynamics of the system may be written as

$$M\ddot{x} + B\dot{x} + K\,x + K'\,x^3 = 0$$

where $Kx + K'\,x^3$ = Opposing force due to nonlinear spring.

- The equation is nonlinear differential equation and it is also called Duffing's equation.

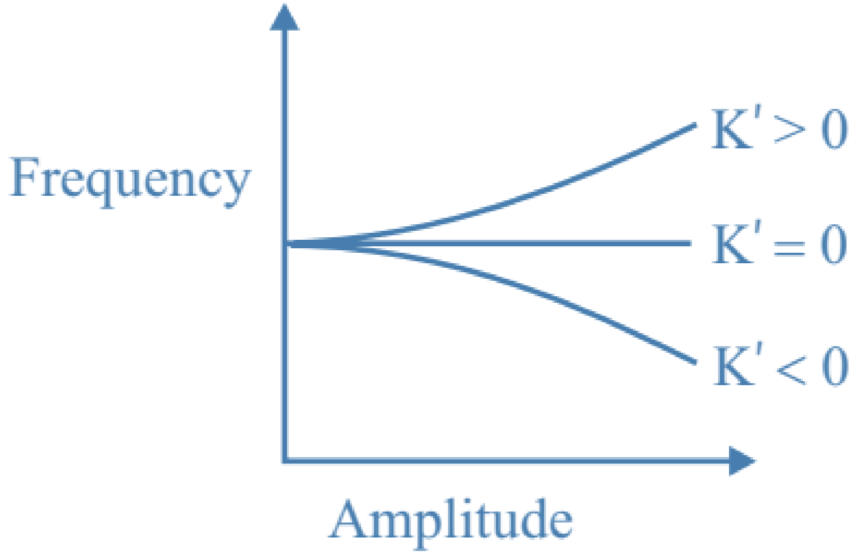


Mechanical system with nonlinear spring

Nonlinear Systems-Characteristics

Frequency -Amplitude Dependence

- The parameters M, B and K are positive constants. The parameter K' may be positive or negative.
- If K' is positive, the spring is called hard spring and if K' is negative the spring is called soft spring.
- When the system has non-zero initial conditions, the free response (i.e., solution of equation) is damped oscillatory.
- The frequency-amplitude dependence characteristic of nonlinear mechanical system is shown in figure

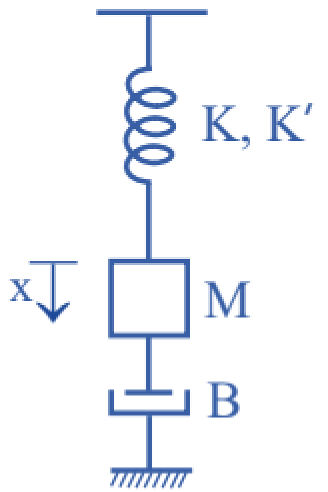


- The frequency of free oscillations depends on the amplitude of oscillations. When K' < 0 (soft spring) the frequency decreases with decreasing amplitude. When K' > 0 (hard spring) the frequency increases with decreasing amplitude. When K' = 0 (corresponding to linear system) the frequency remains unchanged as the amplitude of free oscillation decreases.

Nonlinear Systems-Characteristics-Jump resonance

- In the frequency response of nonlinear systems, the amplitude of the response (output) may jump from one point to another for increasing or decreasing values of frequency, ω.
- This phenomenon is called jump resonance.
- This phenomenon is observed in the frequency response of the system shown in figure when it is subjected to sinusoidal input.
- Let the mechanical system of figure, be subjected to an of type $A\cos\omega t$ (forcing function).
- Now the differential equation governing the mechanical system is

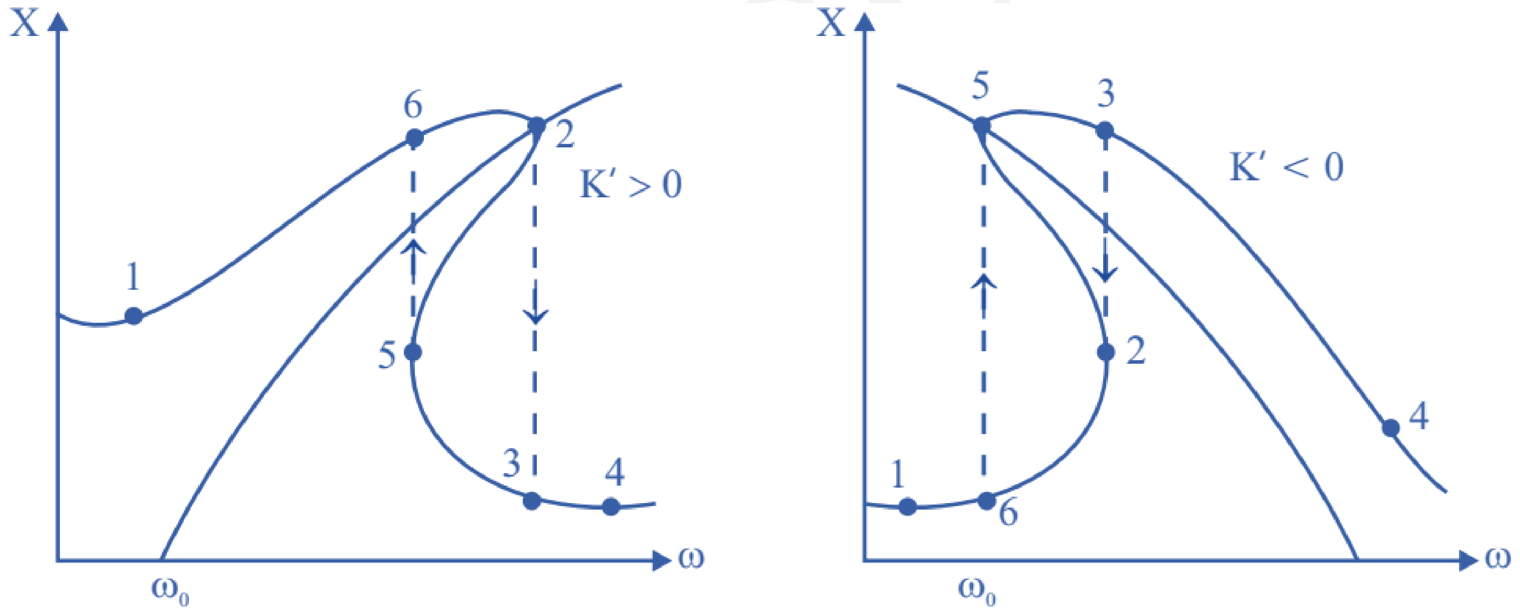
$$M\ddot{x} + B\dot{x} + Kx + K'x^3 = A\cos(\omega t)$$



Mechanical system with nonlinear spring

Nonlinear Systems-Characteristics-Jump resonance

- Let X be the amplitude of the response or output of the system.
- In frequency response studies, the amplitude, A of the input is held constant, while its ω is varied and the amplitude, X of the output is observed.
- The frequency response curve is plotted between X and ω.
- The frequency response curves of the mechanical system of figure are shown in figure a and b for hard and soft springs respectively.

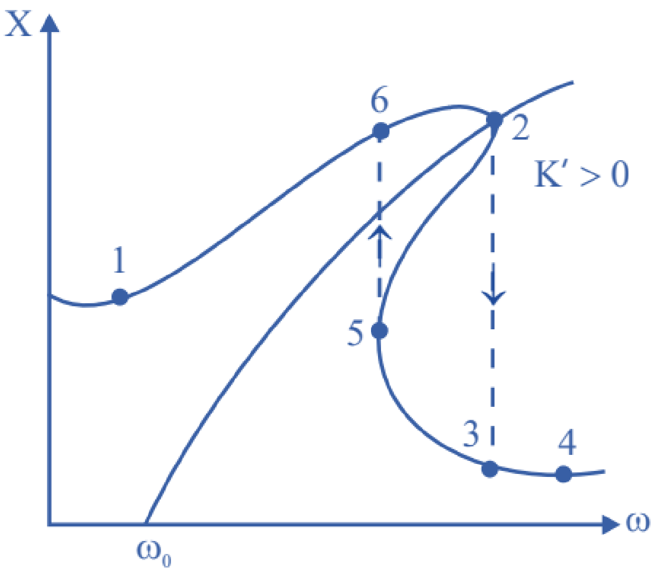


a: Mechanical system with hard spring *b: Mechanical system with soft spring*

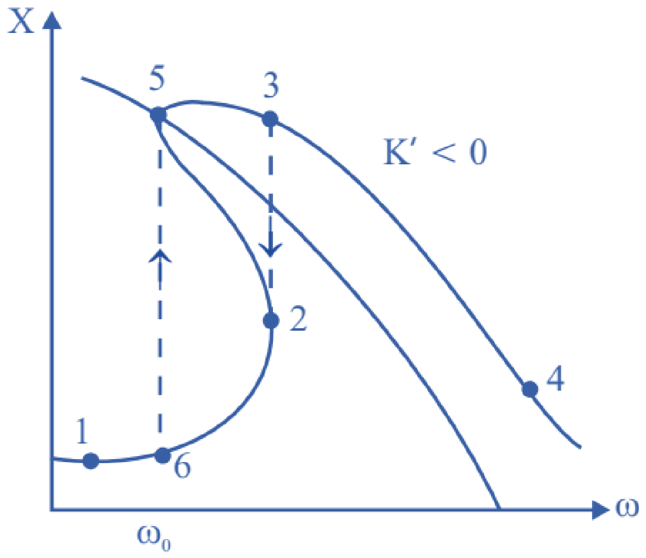
Frequency response curves showing jump resonance.

Nonlinear Systems-Characteristics-Jump resonance

- In the frequency response curve shown in figure a and b, as the frequency ω is increased, the amplitude X increases, until point 2 is reached.
- A further increase in frequency will cause a jump from point 2 to point 3. This phenomenon is called jump resonance.
- As the frequency is increased further, the amplitude X follows the curve from point 3 towards point 4.
- When the frequency is reduced starting from a high value corresponding to point 4, the amplitude X slowly increases through point 3, until point 5 is reached.
- A further decrease in ω will cause another jump from point 5 to point 6. This phenomenon is called jump resonance.
- After this jump, the amplitude X decreases with ω and follows the curve from point 6 towards point 1.
- For jump resonance to take place, it is necessary that the damping term be small and the amplitude of the forcing function be large enough to drive the system into a region of appreciably nonlinear operation.



a: *Mechanical system with hard spring*



b: *Mechanical system with soft spring*

Nonlinear Systems-Characteristics-

Harmonics and sub-harmonic oscillation in the output under sinusoidal input

- Linear systems when excited by a sinusoidal input cannot generate new frequencies at the output as under steady state the frequency of input and output will be the same.
- But for non linear systems, the output may contain several harmonics in addition to fundamental corresponding to input frequency.
- The amplitude of fundamental is usually the largest, but the harmonics may be of significant amplitude in many situations.
- There are also some systems which exhibits sub harmonic oscillations which are of frequencies equal to the integer fractions of the input frequency.

Nonlinear Systems-Characteristics

Limit cycles or sustained oscillations

- Another phenomenon that is observed in certain nonlinear systems is a self-excited oscillation or limit cycle.
- The response (or output) of nonlinear systems may exhibit oscillations with fixed amplitude and frequency. These oscillations are called limit cycles.
- Consider a mechanical system with nonlinear damping and described by the equation,

$$M\ddot{x} - B(1 - x^2)\dot{x} + x = 0$$

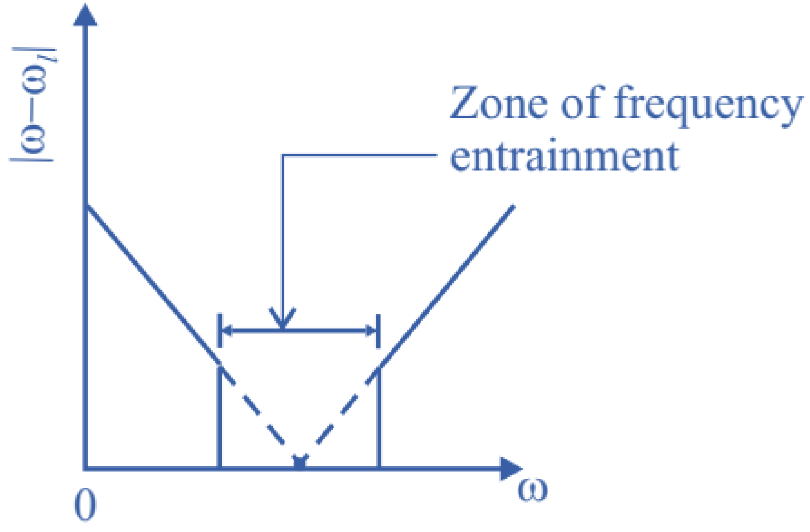
where M, B and K are positive constants.

- The equation is called the van der pol equation.
- For small values of x the damping will be negative which implies the stored energy in the damper is fed to the system.
- For large values of x the damping is positive which implies that it absorbs energy from the system. Thus, it can be expected that such a system may exhibit a sustained oscillation.
- Since the system explained above is not a forced system, this oscillation is called a self-excited oscillation or zero input limit cycle.

Nonlinear Systems-Characteristics-Frequency Entrainment

- The phenomena of frequency entrainment is observed in the frequency response of nonlinear systems that exhibit limit cycles.
- Consider a system capable of exhibiting a limit cycle of frequency ω_l. If a periodic input of frequency ω is applied to this system then the phenomenon of beats is observed.
- The beat is the oscillation whose frequency is the difference between ω_l and ω. This frequency is also called beat frequency.
- In linear systems, the beat frequency decreases indefinitely as ω approaches ω_l.
- But in nonlinear systems, the frequency ω_l of the limit cycle falls in synchronistic ally with or is entrained by the forcing frequency, ω within a certain band of frequencies.

- This phenomenon is called frequency entrainment. The band of frequency in which entrainment occurs is called the zone of frequency entrainment.
- In this zone, the frequencies ω and ω_l coalesce and only one frequency, ω exists. The relationship between $|\omega - \omega_l|$ and ω is shown in figure.



Nonlinear Systems-Characteristics

Asynchronous Quenching

- In a nonlinear system that exhibits a limit cycle of frequency ω_l, it is possible to quench the limit cycle oscillations by forcing the system at a frequency ω_q, where ω_l and ω_q are not related to each other.
- This phenomenon is called asynchronous quenching, or signal stabilization.

Nonlinear Systems-Types

Inherent Nonlinearities or Intentional Nonlinearities

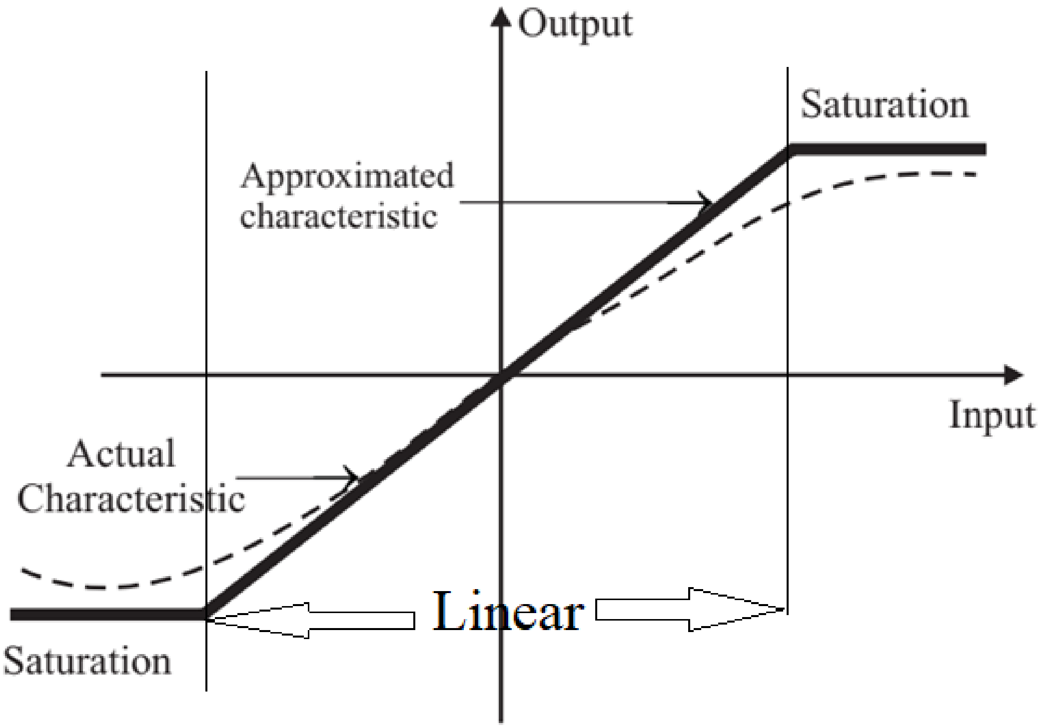
- The incidental or Inherent nonlinearities are those which are inherently present in the system.
- Inherent nonlinearities are unavoidable in control systems.
- Example of such nonlinearities are Saturation, Dead Zone, Hysteresis, Backlash, Friction (Static, Coulomb, etc.), Nonlinear Spring, Compressibility of Fluid etc.
- The intentional nonlinearities are those which are deliberately inserted in the system to modify system characteristics.
- Example of such nonlinearities are Relay-On-off control, Solid state Switches etc.

Nonlinear Systems-Common physical non linearities

Saturation

- In this type of nonlinearity the output is proportional to input for a limited range of input signals. When the input exceeds this range, the output tends to become nearly constant as shown in figure.
- When the input is small, its increase leads to a corresponding (often proportional) increase of output, but when the input reaches a certain level, its further increase does produce little or no increase of the output. The output simply stays around its maximum value. The device is said to be in saturation when this happen.

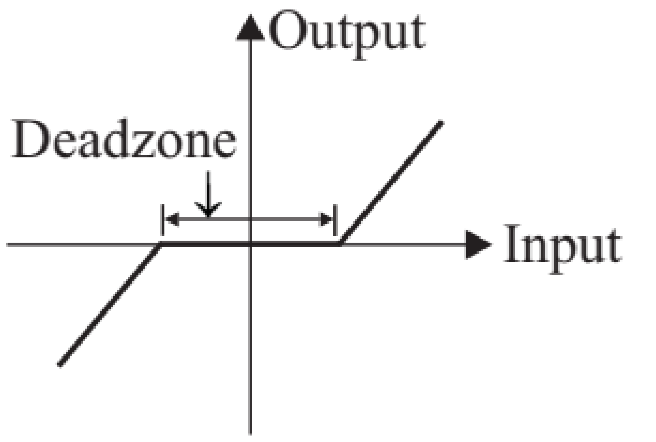
- Almost all devices when driven by sufficiently large signals, exhibit the phenomenon of saturation due to limitations of their physical capabilities.
- Saturation in the output of electronic, rotating and flow (hydraulic and pneumatic) amplifiers, speed and torque saturation in electric and hydraulic motors, saturation in the output of sensors for measuring position, velocity, temperature, etc., are the well known examples.



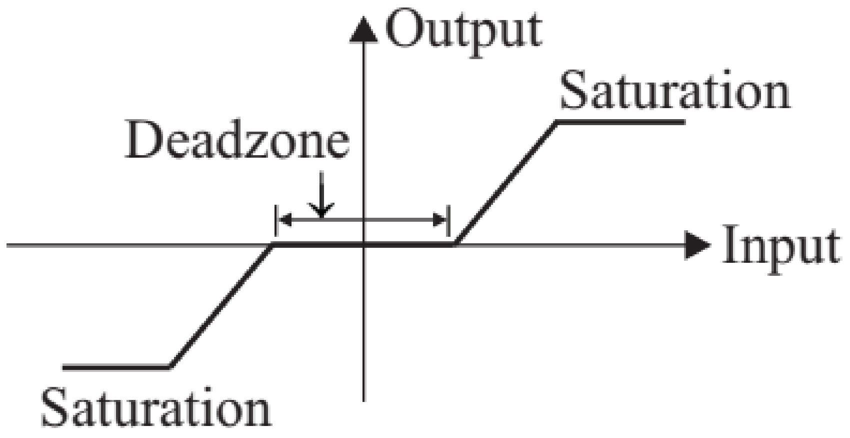
Nonlinear Systems-Common physical non linearities

Dead zone

- The dead zone is the region in which the output is zero for a given input.
- Many physical devices do not respond to small signals, i.e., if the input amplitude is less than some small value, there will be no output.
- The region in which the output is zero is called dead zone.
- When the input is increased beyond this dead zone value, the output will be linear.



Dead zone nonlinearity.

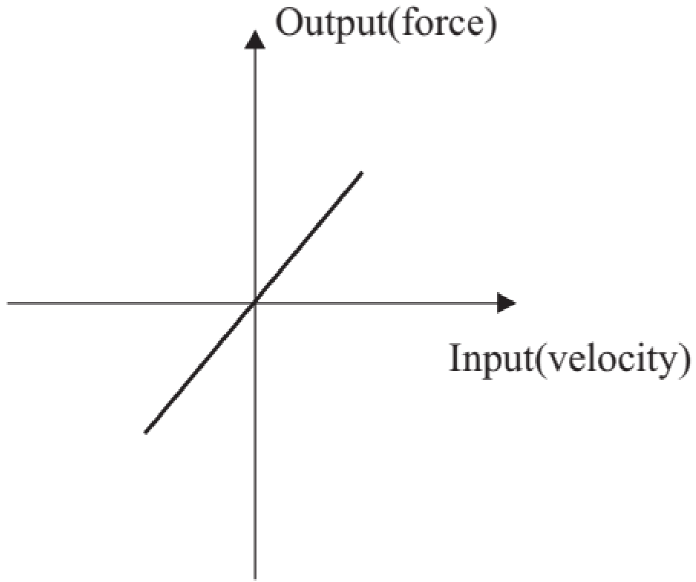


Dead zone and saturation nonlinearity.

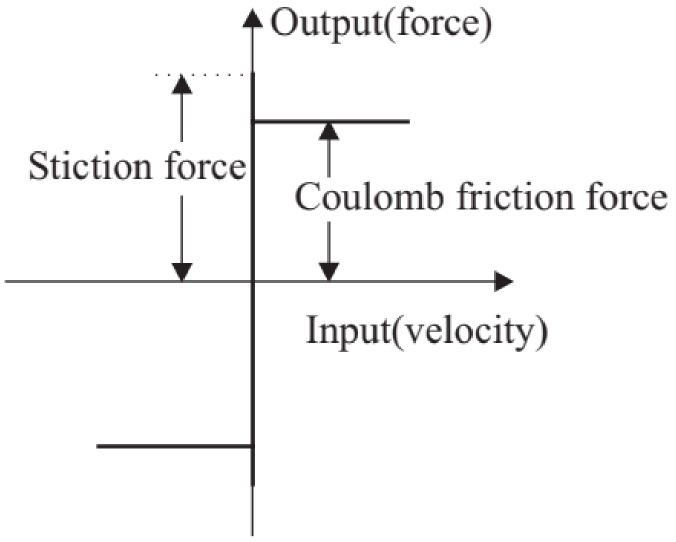
Nonlinear Systems-Common physical non linearities

Friction

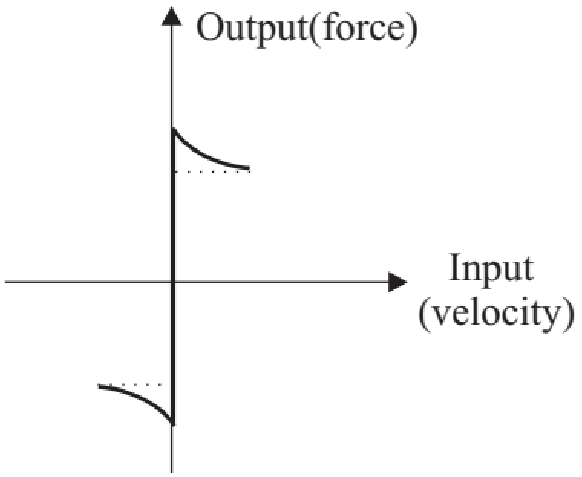
- Friction exists in any system when there is relative motion between contacting surfaces.
- The different types of friction are viscous friction, coulomb friction and stiction.
- The viscous friction is linear in nature and the frictional force is directly proportional to relative velocity of the sliding surfaces.
- The coulomb friction and stiction are nonlinear frictions. The coulomb friction offers a constant retarding force only when the motion is initiated. Due to interlocking of surface irregularities, more force is required to move an object from rest than to maintain it in motion. Hence the force of stiction is always greater than that of coulomb friction.
- In actual practice, the stiction force gradually decreases with velocity and changes over to coulomb friction at reasonably low velocities.



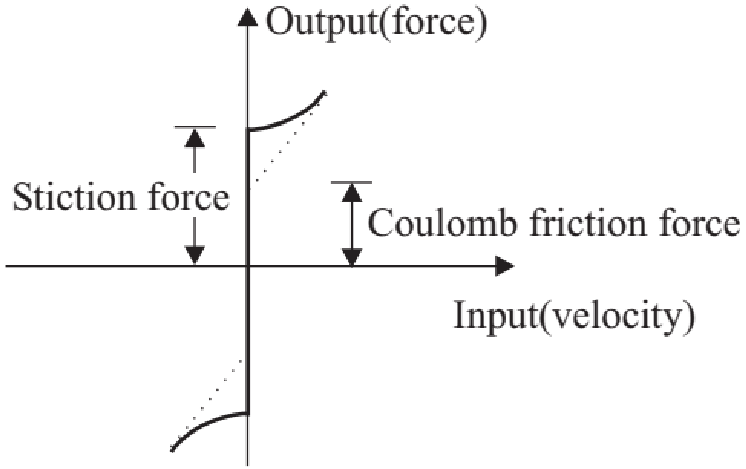
Viscous friction.



Ideal stiction and coulomb friction.



Actual stiction and coulomb friction.

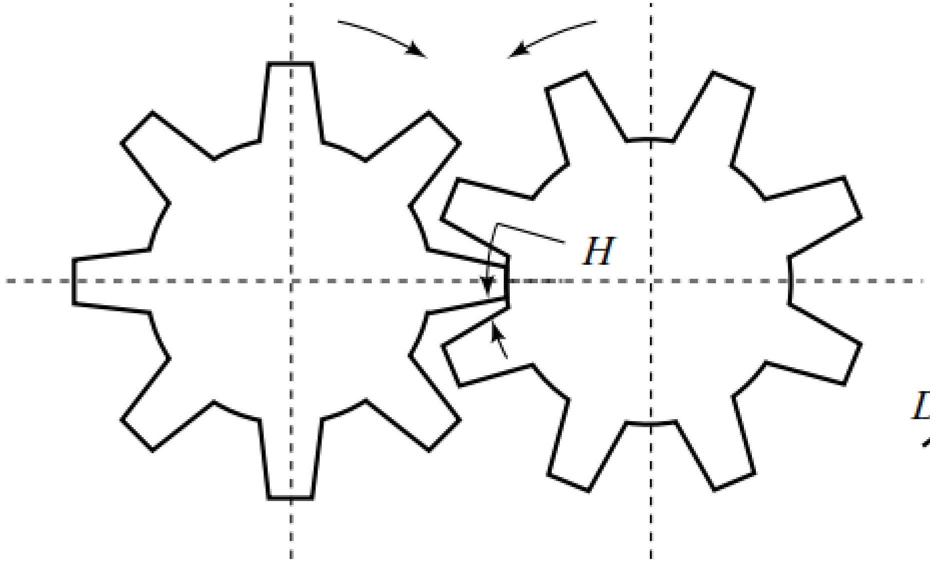


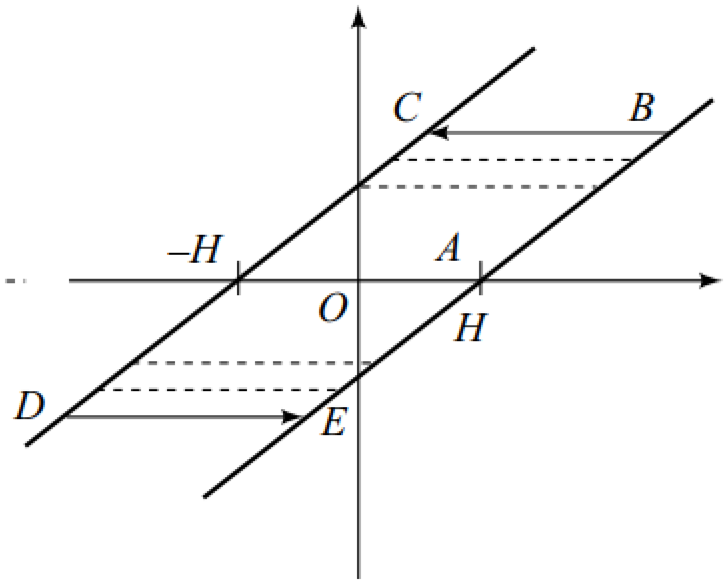
Stiction, coulomb friction and viscous friction.

Nonlinear Systems-Common physical non linearities

Backlash

- Backlash often occurs in transmission systems.
- It is caused by the small gaps which exist in transmission mechanism.
- In gear trains, there always exists small gaps between a pair of mating gears as shown in Figure below





Backlash

- The backlash occurs as result of the unavoidable errors in manufacturing and assembly.
- As a results of the gaps, when the driving gear rotates a smaller angle than the gap H, the driven gear does not move at all, which corresponds to the dead zone (OA segment).
- After contact has been established between the two gears, the driven gear follows the rotation of the driving gear in a linear fashion (AB segment).
- When the driving gear rotates in the reverse direction by a distance of 2H, the driven gear again does not move, corresponding the BC segment.
- After the contact has been re-established between the two gears, the driven gear follows the rotation of the driving gear in a linear fashion in reverse direction (CD segment).
- Therefore, if the driving gear is in periodic motion, the driven gear will move in the fashion represented by the closed path EBCD.
- A critical feature of backlash a form of hysteresis, is its mutlivalued nature. Corresponding to each input, two output values are possible.

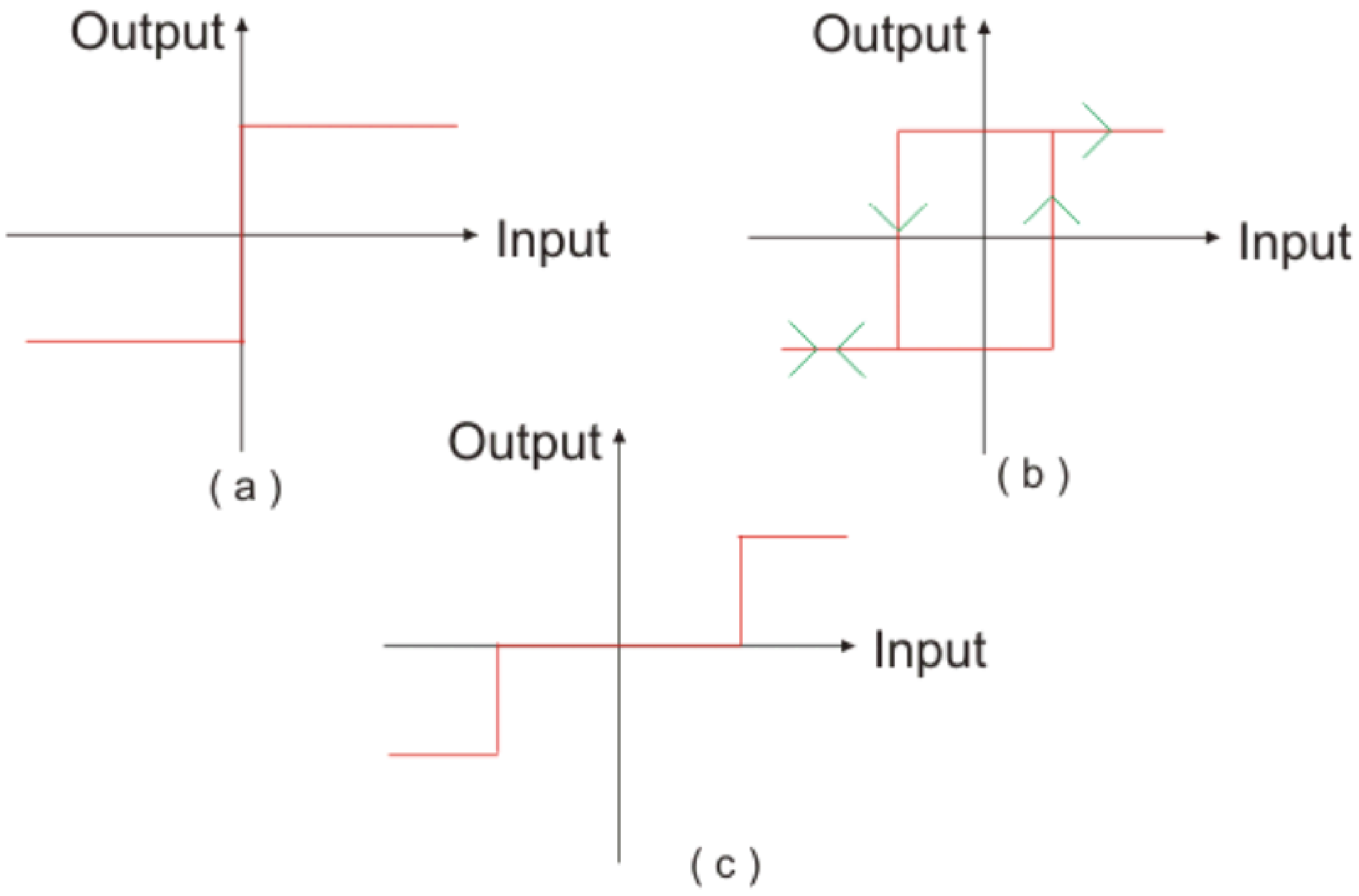
Nonlinear Systems-Common physical non linearities

Relay

- A relay is a nonlinear power amplifier which can provide large power amplification inexpensively and is therefore deliberately introduced in control systems.
- A relay controlled system can be switched abruptly between several discrete states which are usually off, full forward and full reverse.
- Relay controlled systems find wide applications in the control field.
- The characteristic of an ideal relay is as shown in figure.
- In practice a relay has a definite amount of dead zone as shown.
- This dead zone is caused by the facts that relay coil requires a finite amount of current to actuate the relay.
- Further, since a larger coil current is needed to close the relay than the current at which the relay drops out, the characteristic always exhibits hysteresis.

Nonlinear Systems-Common physical non linearities

Relay Non-Linearity (a) ON/OFF (b) ON/OFF with Hysteresis (c) ON/OFF with Dead Zone.



Nonlinear Systems-Types

Static and dynamic non linearity

- Static non linearities are those where the present output of the nonlinear elements depends only on the present value of the input
- Usually the output of the static nonlinear element is written as $y = f(x)$
- However the dynamic non linearities are characterized by the output being a function of input and its time derivatives i.e. $y = f(x, \dot{x}, \ddot{x}, \ldots..)$

Nonlinear Systems-Types

Functional and Piecewise Linear

- The output of some non linear elements can be expressed as a simple function of the input, through nonlinear functions.
- They are called functional type nonlinearity. eg $y = k_1 x + k_2 x^3$.
- There are many commonly occurring non linearities whose output cannot be easily expressed as a mathematical function of input.
- At most they can be represented by piece wise linear approximation. They belong to piecewise non linearities.

Nonlinear Systems-Types

Memory type and memory less

- If the output of the nonlinear element will take one of the several values of the output depending on the past history of the system ,then its called memory type non-linearity.
- If the output of the system depends only on current value of input, then its called memory less nonlinearities.

Describing Function of Nonlinearities

Investigation of Nonlinear Systems

- For analysis, the nonlinear system can be approximated by a linear model in the entire operating region.
- The nonlinear systems can be piecewise approximated. Each piece can be analyzed by a differential equation governing the systems.
- The two popular methods of analyzing nonlinear systems are phase-plane method and describing function method.
- The phase plane method is basically a graphical method from which information about transient behaviour and stability is easily obtained by constructing phase trajectories.
- This method is restricted to second order systems. Higher order systems may first be approximated by their second-order equivalent for investigation by the phase plane method.

Investigation of Nonlinear Systems

- The Describing function method is based on harmonic linearization.
- Here the input to nonlinear component is sinusoidal and depending upon the filtering properties of the linear part of the overall system, the output is adequately represented by the fundamental frequency term in Fourier series.
- The phase-plane and describing function methods use complimentary approximations.
- The phase plane method retains, the nonlinearity as such and uses the second-order approximation of a higher-order linear part, while on the other hand, the describing function method retains the linear part and harmonically linearizes the nonlinearity.

Investigation of Nonlinear Systems

Linearization Techniques

Linearisation of a nonlinear phenomenon near an operating point and studying the behaviour of the system using the linearised model is a commonly used technique for the study of nonlinear systems.

The validity of such a linearised analysis depends on how well the linearised model can represent the actual nonlinear system.

In reality all systems are nonlinear and linear systems are only approximations of the nonlinear systems.

In some cases, the linearisation yields useful information whereas in some other cases, linearised model has to be modified when the operating point moves from one to another.

These are also called by different names as 'perturbation method', series approximation techniques, quasi-linearisation techniques etc, the differences between these being very minor.

Investigation of Nonlinear Systems

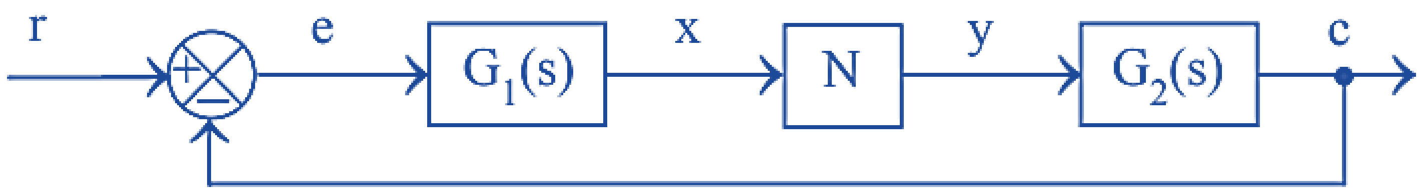
Lyapunov Method for stability

One of the most useful methods for the study of stability of equilibrium points of a nonlinear system is the second method due to A M. Lyapunov which allows one to conclude about the stability without solving the system equations.

Since solving nonlinear system of equations usually requires numerical techniques, Lyapunov method can be used with relative ease to study stability of linear, nonlinear and time varying systems.

Describing Function

- This method is based on harmonic linearization.
- The input to the nonlinear component is sinusoidal and depending upon the filtering properties of the linear part of the overall system, the output is adequately represented by the fundamental frequency.
- Consider the block diagram of the nonlinear system shown in figure.



- In the above system the blocks G_1(s) and G_2(s) (s) represents linear elements and the block N represent nonlinear element.
- Let x = X sinωt be the input to nonlinear element.
- Now the output y of the nonlinear element will be in general a non-sinusoidal periodic function.

Describing Function

- The Fourier series representation of the output y can be expressed as (by assuming that the nonlinearity does not generate subharmonics)

$$y = A_0 + A_1 \sin \omega t + B_1 \cos \omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + ...$$

If the nonlinearity is symmetrical the average value of y is zero and hence the output y is given by

$$y = A_1 \sin \omega t + B_1 \cos \omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + ...$$

- In the absence of an external input (i.e., when r = 0)the output y of the nonlinearity N is feedback to its input through the linear elements G_2(s) and G_1 (s) in tandem.

Describing Function

- If $G_1(s)\ G_2(s)$ has low-pass characteristics, then all the harmonics of y are filtered, so that the input x to the nonlinear element N is mainly contributed by the fundamental component of y and hence x remains sinusoidal.
- Under such conditions the harmonics of the output are neglected and the fundamental component of y alone considered for the purpose of analysis.

$$\therefore\ y = y_1 = A_1 \sin \omega t + B_1 \cos \omega t = Y_1 \angle \phi_1 = Y_1 \sin(\omega t + \phi_1)$$

where, $Y_1 = \sqrt{A_1^2 + B_1^2}$

and $\phi_1 = \tan^{-1} \dfrac{B_1}{A_1}$

Describing Function

Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The coefficients A_1 and B_1 of the fourier series are given by

$$A_1 = \frac{2}{2\pi} \int_0^{2\pi} y \sin \omega t \, d(\omega t)$$

$$B_1 = \frac{2}{2\pi} \int_0^{2\pi} y \cos \omega t \, d(\omega t)$$

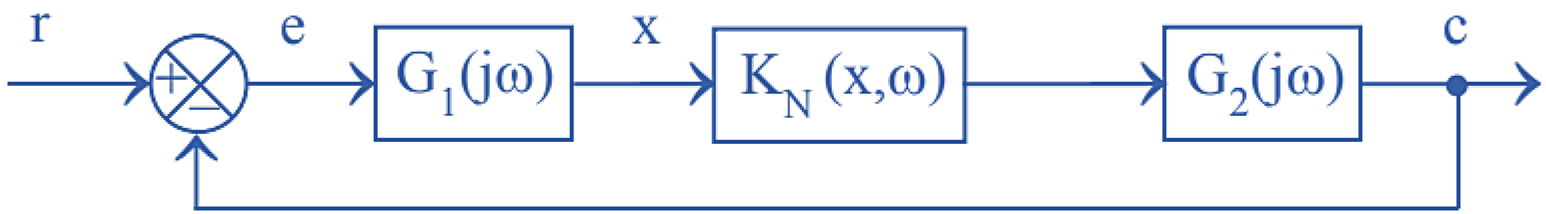
When the input, x to the nonlinearity is sinusoidal (i.e., $x = X \sin \omega t$) the describing function of the nonlinearity is defined as,

$$\mathbf{K_N}(\mathbf{X}, \omega) = \frac{\mathbf{Y_1}}{\mathbf{X}} \angle \phi_1$$

Describing function of a nonlinear element is defined to be the complex ratio of the fundamental harmonic component of the output to the input.

Describing Function

- The nonlinear element N in the system can be replaced by the describing function as shown in figure.



Nonlinear system with nonlinearity replaced by describing function

- If the nonlinearity is replaced by a describing function then all linear theory frequency domain techniques can be used for the analysis of the system.
- The describing functions are used only for stability analysis and it is not directly applied to the optimization of system design.
- The describing function is a frequency domain approach and no general correlation is possible between time and frequency responses.

Describing Function

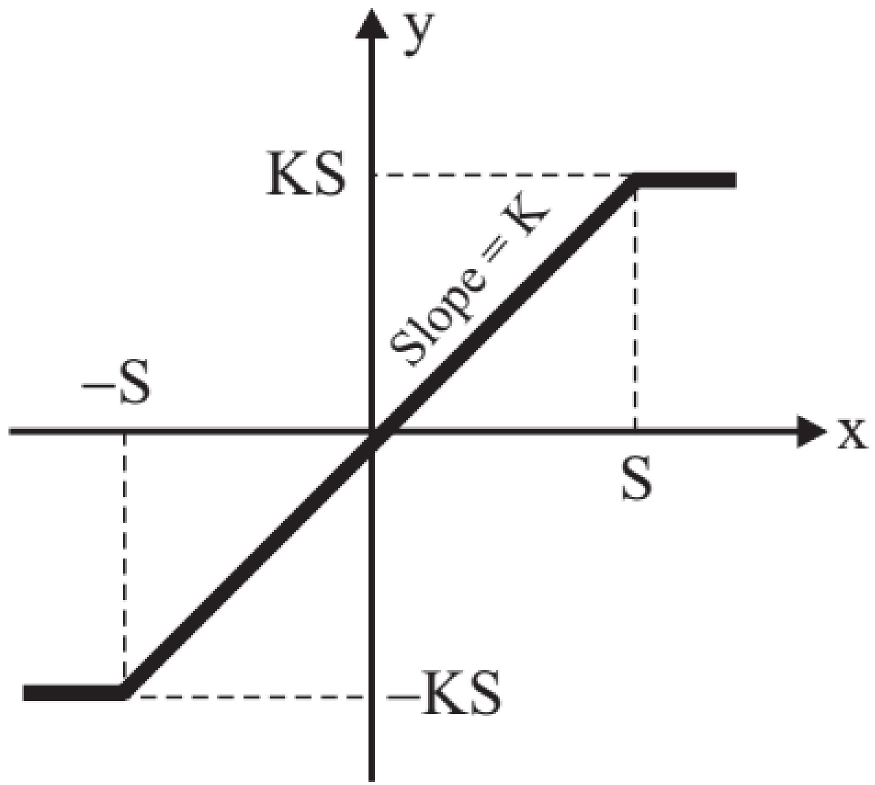
- *Describing Function* $K_N(X,\omega) = \frac{Y_1}{X}\angle\emptyset_1 = \frac{\sqrt{A_1^2+B_1^2}}{X}\angle \tan^{-1}\left(\frac{B_1}{A_1}\right)$

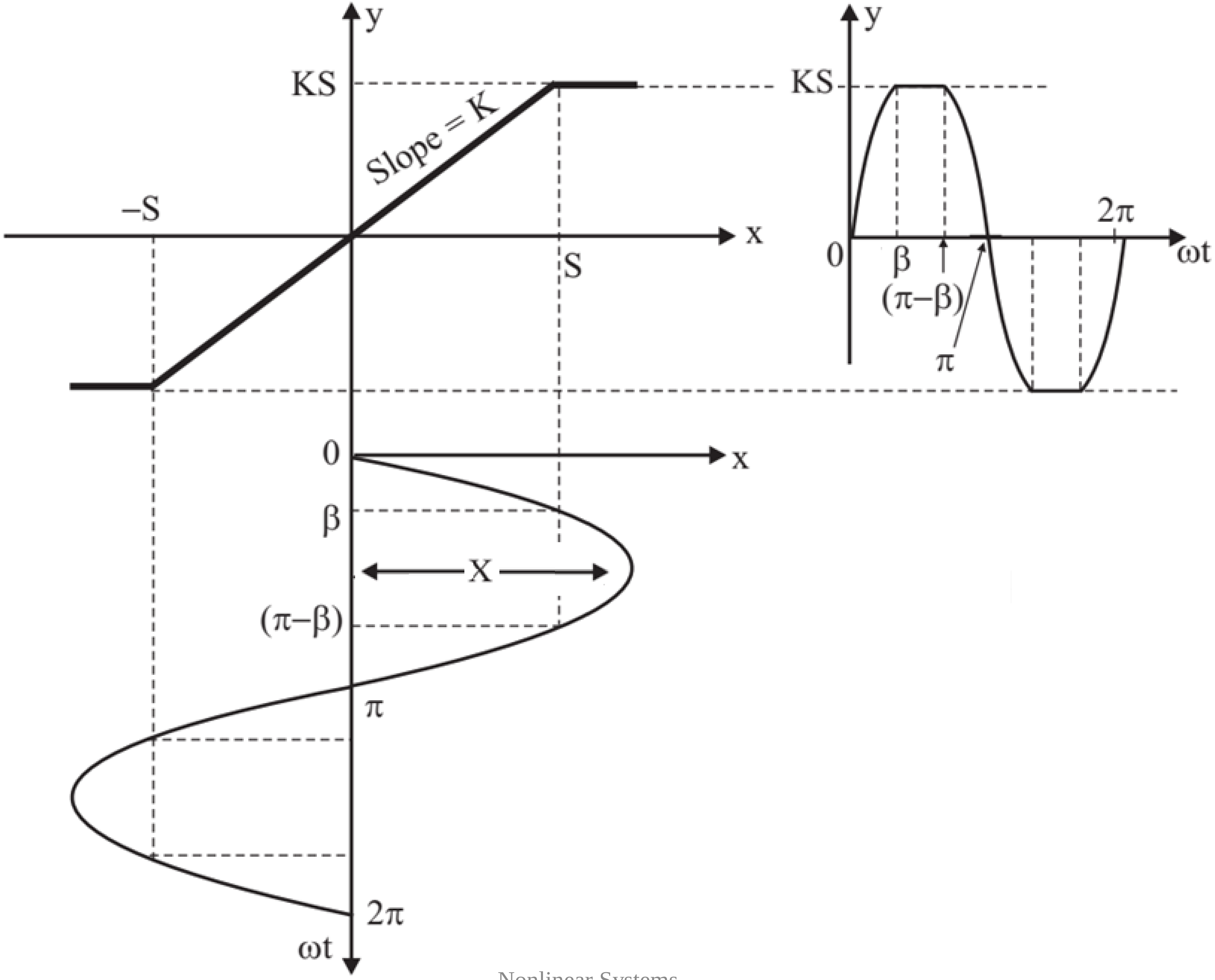
Assumptions used

1. Input to the non linear element should be sinusoidal.
2. Output of the non linear element can be approximated by the fundamental component neglecting the higher order harmonics.
3. The linear plant G(s) should have sufficient low pass filtering characteristics so that the higher frequency components are attenuated significantly while passing around the loop.
4. Linear plant is assumed to be time variant.

Describing Function of Saturation Nonlinearity

- The input-output relationship of saturation nonlinearity is shown in figure.
- The input-output relation is linear for $x = 0$ to S. When the input $x > S$, the output reaches a saturated value of KS.
- The response of the nonlinearity when the input is sinusoidal signal ($x = X \sin \omega t$) is shown in figure below.




y
KS
Slope = K
–S
x
S
0
β
X
(π–β)
π
2π
ωt
y
KS
2π
0
β
(π–β)
π
ωt

Describing Function of Saturation Nonlinearity

The input x is sinusoidal,

$$\therefore\ x = X\sin\omega t \qquad (1)$$

where X is the maximum value of input.

In figure when $\omega t = \beta,\ x = S$.

Hence equation (1) can be written as, $S = X\sin\beta$

$$\therefore \sin\beta = \frac{S}{X} \qquad \text{(or)} \qquad \beta = \sin^{-1}\left(\frac{S}{X}\right)$$

The output y of the nonlinearity can be divided into three regions in a period of π.

The output equation for the three regions are given by equation (2)

$$y = \begin{cases} Kx & ;\ 0 \le \omega t \le \beta \\ KS & ;\ \beta \le \omega t \le (\pi - \beta) \\ Kx & ;\ (\pi - \beta) \le \omega t \le \pi \end{cases} \qquad (2)$$

Describing Function of Saturation Nonlinearity

Let Y_1 = Amplitude of the fundamental harmonic component of the output.

ϕ_1 = Phase shift of the fundamental harmonic component of the output with respect to the input.

The describing function is given by, $K_N(X,\omega) = (Y_1/X) \angle \phi_1$

where $Y_1 = \sqrt{A_1^2 + B_1^2}$ and $\phi_1 = \tan^{-1}(B_1/A_1)$

The output y has half wave and quarter wave symmetries

$$\therefore \; B_1 = 0 \quad \text{and} \quad A_1 = \frac{2}{\pi/2} \int_0^{\frac{\pi}{2}} y \sin\omega t \, d(\omega t)$$

The output, y is given by two different expressions in the period 0 to $\pi/2$. Hence equation can be written as shown in equation

$$A_1 = \frac{2}{\pi/2} \int_0^{\beta} y \sin\omega t \, d(\omega t) + \frac{4}{\pi} \int_{\beta}^{\frac{\pi}{2}} y \sin\omega t \, d(\omega t)$$

$$A_1 = \frac{2}{\pi/2}\int_0^{\beta} y \sin\omega t \, d(\omega t) + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}} y \sin\omega t \, d(\omega t)$$

On substituting the values of y from equation (2) we get,

$$A_1 = \frac{4}{\pi}\int_0^{\beta} Kx \sin\omega t \, d(\omega t) + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}} KS \sin\omega t \, d(\omega t)$$

On substituting x = X sinωt, we get,

$$A_1 = \frac{4K}{\pi}\int_0^{\beta} X \sin\omega t \times \sin\omega t \, d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \, d(\omega t)$$

$$= \frac{4KX}{\pi}\int_0^{\beta} \sin^2\omega t \, d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \, d(\omega t)$$

$$= \frac{4KX}{\pi}\int_0^{\beta} \frac{1-\cos 2\omega t}{2} \, d(\omega t) + \frac{4KS}{\pi}\int_{\beta}^{\frac{\pi}{2}} \sin\omega t \, d(\omega t)$$

$$= \frac{2KX}{\pi}\left[\omega t - \frac{\sin 2\omega t}{2}\right]_0^{\beta} + \frac{4KS}{\pi}\left[-\cos\omega t\right]_{\beta}^{\frac{\pi}{2}}$$

Describing Function of Saturation Nonlinearity

$$= \frac{2KX}{\pi}\left[\beta - \frac{\sin 2\beta}{2}\right] + \frac{4KS}{\pi}\left[-\cos\frac{\pi}{2} + \cos\beta\right]$$

$$= \frac{2KX}{\pi}\left[\beta - \frac{\sin 2\beta}{2}\right] + \frac{4KS}{\pi}\cos\beta$$

On substituting for S, (i.e, S = X sinβ)

$$A_1 = \frac{2KX}{\pi}\left[\beta - \frac{\sin 2\beta}{2}\right] + \frac{4K}{\pi} X \sin\beta\cos\beta$$

$$= \frac{2KX}{\pi}\left[\beta - \frac{2\sin\beta\cos\beta}{2}\right] + \frac{4KX}{\pi}\sin\beta\cos\beta$$

$$= \frac{2KX}{\pi}[\beta - \sin\beta\cos\beta + 2\sin\beta\cos\beta]$$

$$= \frac{2KX}{\pi}[\beta + \sin\beta\cos\beta]$$

Describing Function of Saturation Nonlinearity

$$Y_1 = \sqrt{A_1^2 + B_1^2} = \sqrt{A_1^2 + 0} = A_1 = \frac{2KX}{\pi}[\beta + \sin\beta\cos\beta]$$

$$\phi_1 = \tan^{-1}\frac{B_1}{A_1} = \tan^{-1}0 = 0$$

The describing function, $K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1$

$$\mathbf{K_N(X,\omega) = \frac{Y_1}{X}\angle\phi_1 = \frac{2K}{\pi}[\beta + \sin\beta\cos\beta]\angle 0^\circ}$$

Depending on the maximum value of input X, the describing function can be written as,

If $X < S$, then $\beta = \frac{\pi}{2}$, $\mathbf{K_N(X,\omega) = K}$

If $X > S$, $\mathbf{K_N(X,\omega) = \frac{2K}{\pi}[\beta + \sin\beta\cos\beta]}$

Describing Function of Saturation Nonlinearity

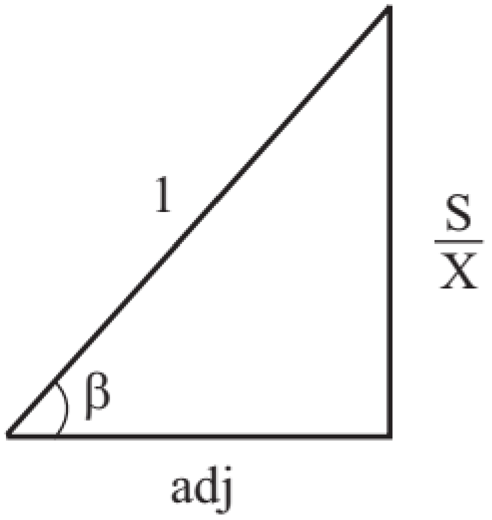
- We have, $S = X \sin\beta,$ $\quad \therefore \sin\beta = \frac{S}{X}$
- On constructing right angle triangle with unity hypotenuse as shown in figure

$$\text{adj} = \sqrt{1-\left(\frac{S}{X}\right)^2} \qquad \therefore \cos\beta = \frac{\text{adj}}{\text{hyp}} = \sqrt{1-\left(\frac{S}{X}\right)^2}$$

- Describing Function

$$\text{If } X > S,\ K_N(X,\omega) = \frac{2K}{\pi}[\beta + \sin\beta\cos\beta]$$

$$\therefore K_N(X,\omega) = \frac{2K}{\pi}\left[\sin^{-1}\left(\frac{S}{X}\right) + \left(\frac{S}{X}\right)\sqrt{1-\left(\frac{S}{X}\right)^2}\right] \text{ for } X > S$$



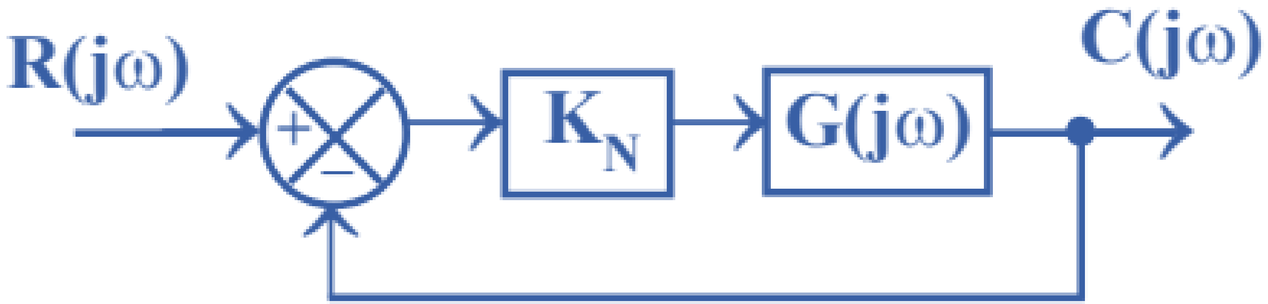
- and $\text{If } X < S, \text{ then } \beta = \frac{\pi}{2},\ K_N(X,\omega) = K$

Describing Function Analysis of Nonlinear Systems

Describing Function Analysis of Nonlinear Systems

The describing functions of nonlinear elements can be used

- For stability analysis of nonlinear control systems.
- To predict the sustained oscillations or limit cycles in the output of the system.
- Consider a unity feedback system shown in figure in which the nonlinearity is represented by its describing function $K_N(X, \omega)$ or K_N.



Describing Function Analysis of Nonlinear Systems

- Let C(jω)/ R(jω) be the closed loop sinusoidal transfer function of the system shown in figure

$$\frac{C(j\omega)}{R(j\omega)} = \frac{K_N G(j\omega)}{1 + K_N G(j\omega)}$$

- The characteristic equation of the system is obtained by equating the denominator of equation to zero. Hence the characteristic equation is given by,

$$1 + K_N G(j\omega) = 0$$

- The Nyquist stability criterion can also be extended to the stability analysis of nonlinear systems.
- According to the Nyquist stability criterion the system will exhibit sustained oscillations or limit cycles when,

$$K_N G(j\omega) = -1$$

Describing Function Analysis of Nonlinear Systems

$$K_N G(j\omega) = -1$$

- The above equation implies that the sustained oscillations or limit cycles will occur if $K_N\ G(j\omega)$ locus pass through the critical point, $-1 + j0$, in the complex plane.
- The equation can be modified as shown below

$$G(j\omega) = -\frac{1}{K_N}$$

- The equation implies that the critical point, $-1 + j0$ becomes the critical locus which is the locus of $-{}^{1}/_{K_N}$. Hence the intersection point of $G(j\omega)$ locus and $-{}^{1}/_{K_N}$ locus will give the amplitude and frequency of limit cycles.

Describing Function Analysis of Nonlinear Systems

- In the stability analysis, let us assume that the linear part of the system is stable.
- To determine the stability of the system due to nonlinearity sketch the $-{}^{1}/_{K_N}$ locus and $G(j\omega)$ locus (polar plot of $G(j\omega)$) in complex plane and from the sketches the following conclusions can be obtained.

1. If the $-{}^{1}/_{K_N}$ locus is not enclosed by the $G(j\omega)$ locus then the system is stable or there is no limit cycle at steady state.
2. If the $-{}^{1}/_{K_N}$ locus is enclosed by the $G(j\omega)$ locus then the system is unstable.
3. If the $-{}^{1}/_{K_N}$ locus and the $G(j\omega)$ locus intersect, then the system output may exhibit a sustained oscillation or a limit cycle. The amplitude of the limit cycle is given by the value of $-{}^{1}/_{K_N}$ locus at the intersection point. The frequency of the limit cycle is given by the frequency of $G(j\omega)$ corresponding to the intersection point.

(Use either a polar graph sheet or ordinary graph sheet)

Concept of Enclosure

- In a complex plane the $-{}^{1}/_{K_N}$ locus is said to be enclosed by G(jω) locus if it lies in the region to the right of an observer travelling through G(jω) locus in the direction of increasing ω
- In a complex plane $-{}^{1}/_{K_N}$ locus is not enclosed by G(jω) if it lies in the region to the left of an observer travelling through G(jω) locus in the direction of increasing ω

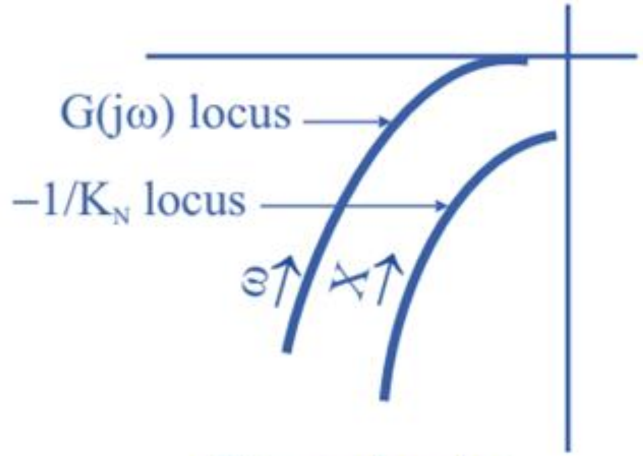


Figure showing enclosure of $-1/K_N$ locus by G(jω) locus.

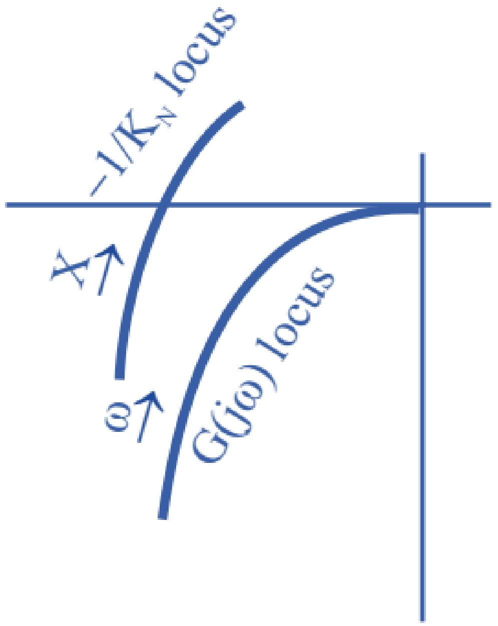


Figure showing nonenclosure of $-1/K_N$ locus by G(jω) locus.

Stable and Unstable Region

- If the $-{}^{1}/_{K_N}$ locus and $G(j\omega)$ locus intersect, then for an observer travelling through $G(j\omega)$ locus in the direction of increasing ω, the region on the right is unstable region and the region on the left side is stable

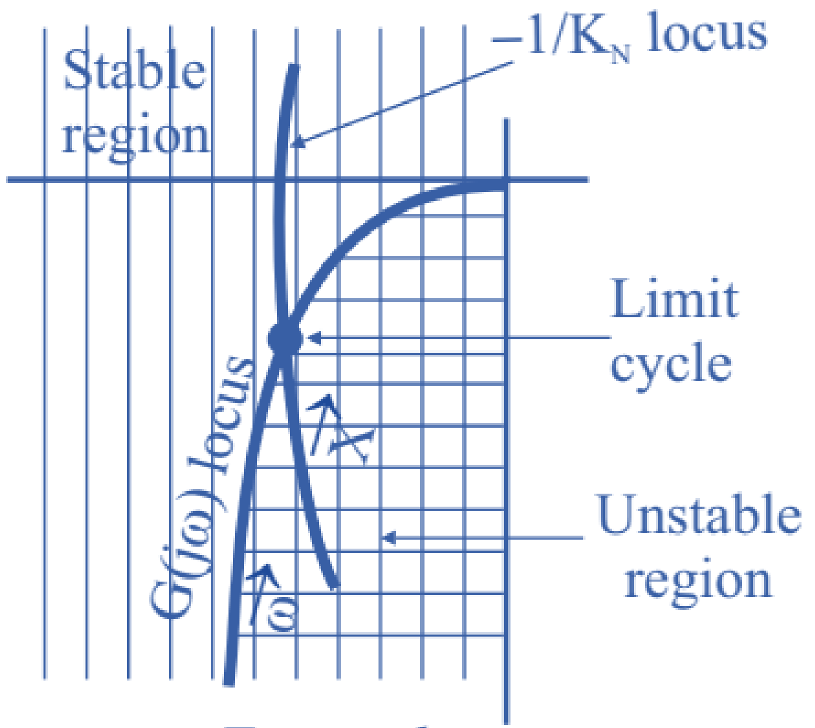
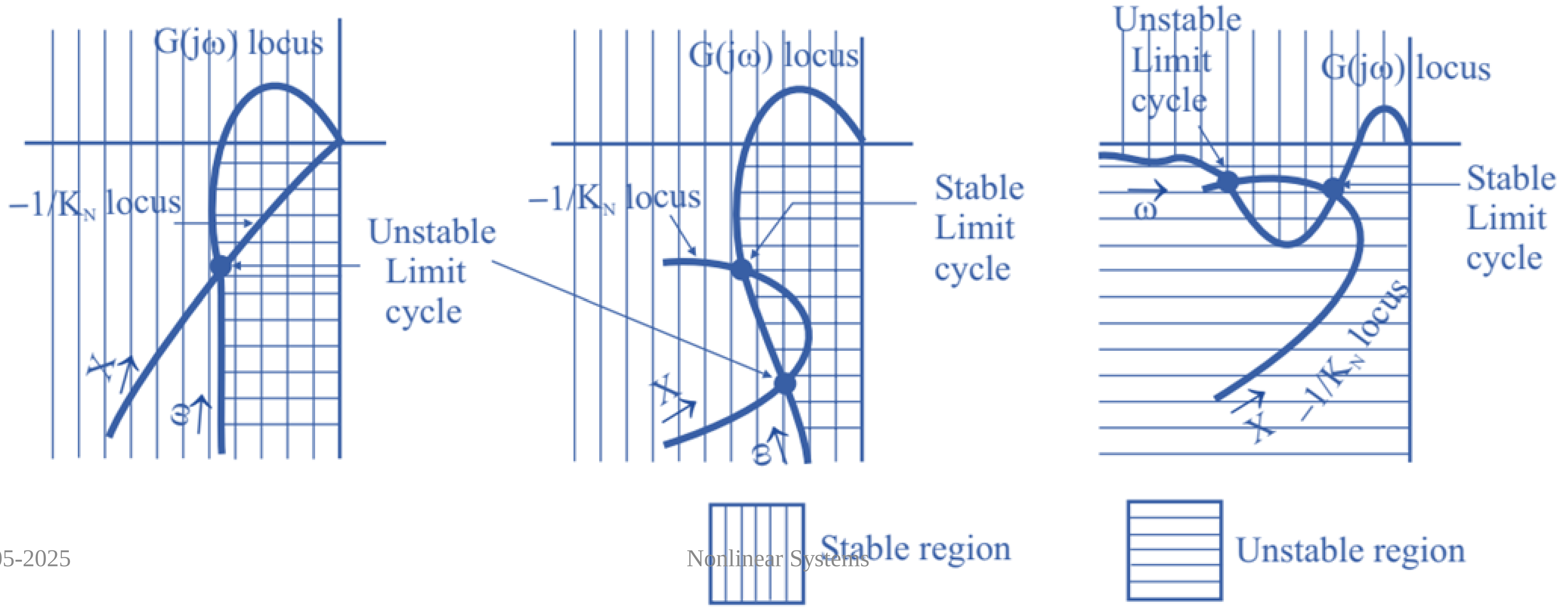


Figure showing intersection of $-1/K_N$ locus by G(jω) locus.

Stable and Unstable Limit Cycles

- The $-{}^{1}/_{K_N}$ locus may intersect $G(j\omega)$ locus at one or more points. There exists a limit cycle at every intersecting point. These limit cycles can be either stable or unstable limit cycles, as shown in figure.
- If $-{}^{1}/_{K_N}$ locus travels in unstable region and it intersect $G(j\omega)$ locus to enter stable region then the limit cycle corresponding to that intersection point is stable limit cycle.
- If $-{}^{1}/_{K_N}$ locus travels in stable region and it intersect $G(j\omega)$ locus to enter unstable region then the limit cycle corresponding to that intersection point is unstable limit cycle.

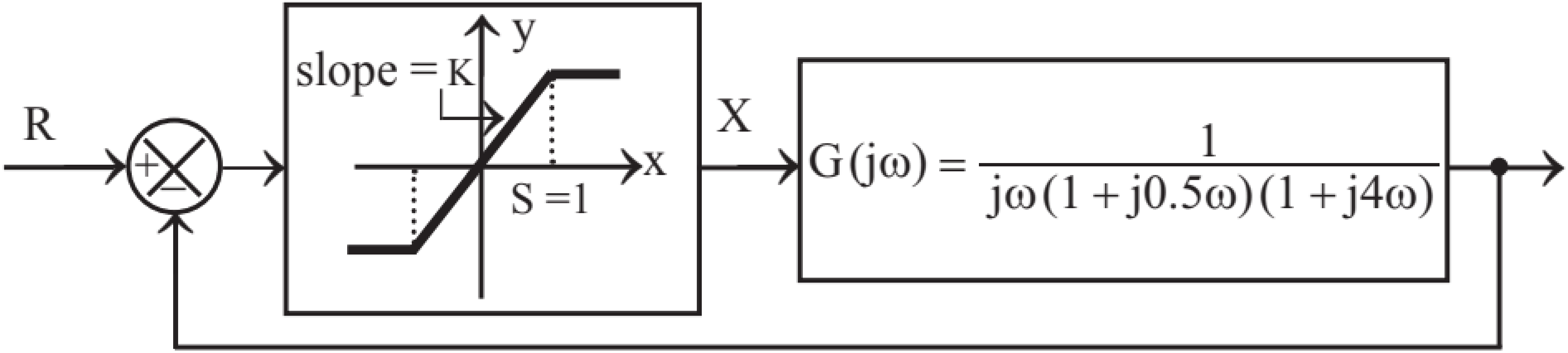


Existence of limit cycles and determination of amplitude and frequency

- Given the non linear element and the linear plant, derive the describing function $K_N(X)$for non linear element.
- Plot $\frac{-1}{K_N(X)}$ locus for X increasing from zero to a large values and mark the direction of increasing X.
- Plot $G(j\omega)$ locus for ω varying from 0 to infinity and mark the direction of increasing ω
- Check whether there is intersection of the $\frac{-1}{K_N(X)}$ locus and $G(j\omega)$locus. If there is any intersection, then limit cycle exits and if there is no intersection, then there is no limit cycles.
- Determine the amplitude X from $\frac{-1}{K_N(X)}$ locus and frequency ω from $G(j\omega)$ locus
- Check the stability of limit cycle.

Describing Function Analysis of Nonlinear Systems-Problem

- Consider a unity feedback system shown in figure having a saturating amplifier with gain K. Determine the maximum value of K for the system to stay stable. What would be the frequency and nature of limit cycle for a gain of K = 2.5?



Describing Function Analysis of Nonlinear Systems-Problem

SOLUTION

The stability of the system can be analysed using polar plot. The gain, K of the saturating amplifier can be attached to G(jω) and amplifier is considered to be an unity gain amplifier.

Polar plot of G(jω) when K = 1

$$\text{Here,}\quad G(j\omega) = \frac{K}{j\omega(1 + j0.5\omega)(1 + j4\omega)}$$

$$\text{Let,}\quad K = 1, \ \therefore G(j\omega) = \frac{1}{j\omega(1 + j0.5\omega)(1 + j4\omega)}$$

$$= \frac{1}{\omega\angle 90^{o}\sqrt{1 + 0.25\omega^2}\angle\tan^{-1}0.5\omega\sqrt{1 + 16\omega^2}\angle\tan^{-1}4\omega}$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1 + 0.25\omega^2}\sqrt{1 + 16\omega^2}}$$

$$\angle G(j\omega) = -90^{o} - \tan^{-1}0.5\omega - \tan^{-1}4\omega$$

- The magnitude and phase of G(jω) are calculated for various values of ω and listed in table.
- Using polar to rectangular conversion the real part and imaginary part of G(jω) are determined and listed in table.
- The polar plot of G(jω) is sketched in an ordinary graph sheet.

ω rad/sec	0.4	0.5	0.6	0.8	1. 0	1.2
$\lvert G(j\omega)\rvert$	1.299	0.868	0.614	0.346	0.216	0.145
$\angle G(j\omega)$	–159°	–167°	–174°	–184°	–192°	–199°
$G_R(j\omega)$	–1.21	–0.85	–0.61	–0.35	–0.21	–0.14
$G_I(j\omega)$	–0.47	–0.2	–0.06	0.02	0.04	0.05

Polar plot of G(jω) when K = 2.5

$$\text{When } K = 2.5, \quad |G(j\omega)| = \frac{2.5}{\omega\sqrt{1+0.5\omega^2}\sqrt{1+16\omega^2}}$$

- The phase of G(jω) is not altered by the term, K.
- The magnitude and phase of G(jω) when K = 2.5 are calculated for various values of ω and listed in table.
- Using polar to rectangular conversion the real part and imaginary part of G(jω) when K = 2.5 are determined and listed in table.
- The polar plot of G(jω) when K = 2.5 is sketched in the same graph sheet using the same scales,

ω rad/sec	0.6	0.65	0.75	0.8	1.0	1.2
$\lvert G(j\omega)\rvert$	1.535	1.313	0.987	0.865	0.54	0.363
$\angle G(j\omega)$	–174	–177	–182	–184	–192	–199
$G_R(j\omega)$	–1.52	–1.31	–0.99	–0.87	–0.53	–0.34
$G_I(j\omega)$	–0.16	–0.07	0.03	0.06	0.11	0.12

Polar plot of $-1/K_N$

The function $-1/K_N$ can be expressed as,

$$\frac{-1}{K_N} = -1 \times \frac{1}{K_N} = 1\angle -180^o \times \frac{1}{K_N}$$

We know that the describing function (K_N) of saturation nonlinearity is given by

$$K_N = \begin{cases} K & ;\ \text{when } X < S \qquad (\because K = 1) \\ \dfrac{2K}{\pi}(\beta + \sin\beta\cos\beta)\angle 0^o & ;\ \text{when } X > S \end{cases}$$

where, $\beta = \sin^{-1}(S/X)$

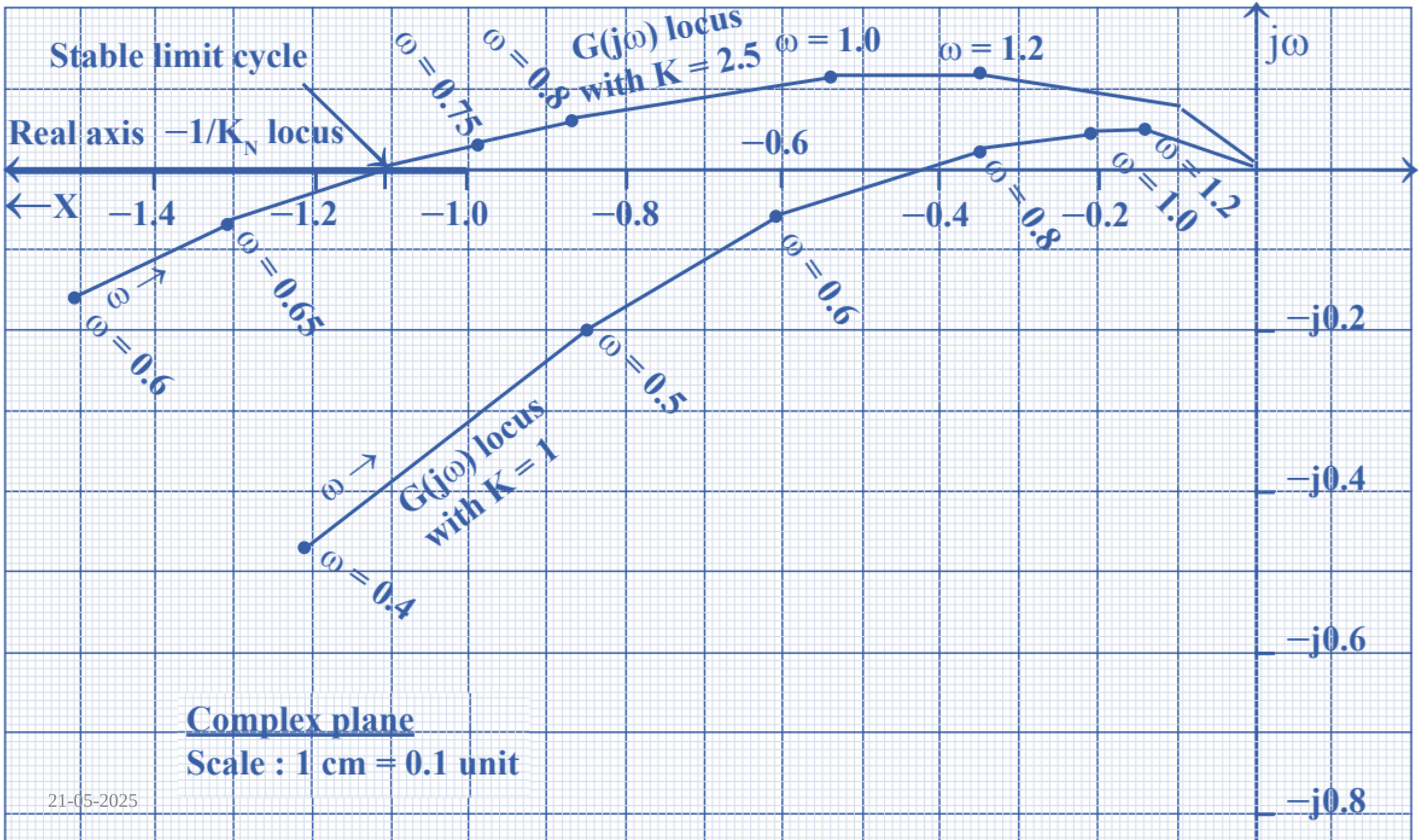
and X = Maximum value of input sinusoidal signal

Here, $K = 1$ and $S = 1$

$$\therefore -1/K_N = \begin{cases} 1\angle -180^\circ & ; \text{ when } X < 1 \\ \dfrac{\pi}{2(\beta + \sin\beta\cos\beta)}\angle -180^\circ & ; \text{ when } X > 1 \end{cases}$$

where, $\beta = \sin^{-1}(1/X)$

- From the equation of $-{}^{1}/_{K_N}$, the locus of $-{}^{1}/_{K_N}$ starts at $1\angle -180°$ (i.e., $-1+j0$) and travels along the negative real axis for increasing values of X as shown in figure.
- The locus of $-{}^{1}/_{K_N}$ is shown as a bold line on the negative real axis.


Stable limit cycle
Real axis −1/K_N locus
←X
−1.4
−1.2
−1.0
−0.8
−0.6
−0.4
−0.2
jω
−j0.2
−j0.4
−j0.6
−j0.8
G(jω) locus with K = 2.5
ω = 0.6
ω = 0.65
ω = 0.75
ω = 0.8
ω = 1.0
ω = 1.2
G(jω) locus with K = 1
ω = 0.4
ω = 0.5
ω = 0.6
ω = 0.8
ω = 1.0
ω = 1.2
ω →
ω →
Complex plane
Scale : 1 cm = 0.1 unit

STABILITY ANALYSIS

Case (i) when K = 1

When K = 1, the G(jω) locus does not encloses the $-1/K_N$ locus, hence the system is stable.

Case(ii) To find maximum value of K for stability

When K is increased the G(jω) locus shifts upwards. For a particular value of K, the G(jω) locus crosses the starting point (i.e., $-1+j0$) of $-1/K_N$ locus and this value of K is the limiting value of K for stability.

If G(jω) crosses negative real axis at $-1+j0$, then, $G(j\omega) = -1 = 1\angle{-180°}$

$$\therefore |G(j\omega)| = 1 \text{ and } \angle G(j\omega) = -180°$$

Let, ω_{l1} = Frequency when $G(j\omega) = -1$

$$\therefore \text{At } \omega = \omega_{l1}, \angle G(j\omega) = -90° - \tan^{-1} 0.5\omega_{l1} - \tan^{-1} 4\omega_{l1} = -180°$$

$$\therefore \tan^{-1} 0.5\omega_{l1} + \tan^{-1} 4\omega_{l1} = 90°$$

On taking tan on either side we get,

$$\tan(\tan^{-1} 0.5\omega_{l1} + \tan^{-1} 4\omega_{l1}) = \tan 90°$$

$$\frac{\tan(\tan^{-1} 0.5\omega_{l1}) + \tan(\tan^{-1} 4\omega_{l1})}{1 - \tan(\tan^{-1} 0.5\omega_{l1}) \times \tan(\tan^{-1} 4\omega_{l1})} = \tan 90°$$

$$\frac{0.5\omega_{l1} + 4\omega_{l1}}{1 - 0.5\omega_{l1} \times 4\omega_{l1}} = \infty$$

For the above equation to be infinity, the denominator should be zero.

$$\therefore\ 1 - 2\omega_{l1}^2 = 0 \quad ; \quad \omega_{l1}^2 = 1/2 \qquad \text{(or)} \qquad \omega_{l1} = \frac{1}{\sqrt{2}} \text{ rad/sec}$$

at $\omega = \omega_{l1}$, $|G(j\omega)| = 1$

$$\therefore \frac{K}{\omega_{l1}\sqrt{1+0.25\omega_{l1}^2}\sqrt{1+16\omega_{l1}^2}}=1 \quad \text{(or)} \quad K=\omega_{l1}\sqrt{1+0.25\omega_{l1}^2}\sqrt{1+16\omega_{l1}^2}$$

$$K=\frac{1}{\sqrt{2}}\sqrt{(1+0.25\times 0.5)(1+16\times 0.5)} = 2.25$$

Therefore the system remains stable if, K < 2.25

Case (iii) when K = 2.5

When K = 2.5, the G(jω) locus intersects, $-1/K_N$ locus at $-1.11+j0$. At the intersection point a stable limit cycle exists.

Coordinate corresponding to stable limit cycle $= -1.11+j0 = 1.11\angle{-180°}$

Let, ω_{l2} = Frequency of stable limit cycle

At $\omega=\omega_{l2}$, $G(j\omega)=1.11\angle{-180°}$

$\therefore$ At $\omega=\omega_{l2}$, $\angle G(j\omega) = -90° - \tan^{-1}0.5\omega_{l2} - \tan^{-1}4\omega_{l2} = -180°$

$$\therefore \tan^{-1} 0.5\omega_{l2} + \tan^{-1} 4\omega_{l2} = 90°$$

On taking tan on either side we get,

$$\tan(\tan^{-1} 0.5\omega_{l2} + \tan^{-1} 4\omega_{l2}) = \tan 90°$$

$$\frac{\tan(\tan^{-1}0.5\omega_{l2}) + \tan(\tan^{-1}4\omega_{l2})}{1 - \tan(\tan^{-1}0.5\omega_{l2}) \times \tan(\tan^{-1}4\omega_{l2})} = \tan 90°$$

$$\frac{0.5\omega_{l2} + 4\omega_{l2}}{1 - 0.5\omega_{l2} \times 4\omega_{l2}} = \infty$$

For the above equation to be infinity, the denominator should be zero.

$$\therefore 1 - 2\omega_{l2}^2 = 0 \quad \text{(or)} \quad \omega_{l2}^2 = 1/2 \text{ rad/sec} \quad \text{(or)} \quad \omega_{l2} = 1/\sqrt{2} \text{ rad/sec}$$

$\therefore$ Frequency of limit cycle = $1/\sqrt{2}$ = 0.707 rad/sec

RESULT

1. When K = 1, the system is stable
2. The system remains stable if K < 2.25
3. When K = 2.5, a stable limit cycle occurs, whose frequency of oscillation is 0.707 rad/sec.

Singular Points and Classifications

Singular Points

- Singular points are points in the state plane where $\dot{x}_1 = \dot{x}_2 = 0$.
- At this point, slope of the trajectory $\frac{dx_2}{dx_1}$ is indeterminate.
- These points can also be the equilibrium points of the non linear systems, depending on whether the state trajectories can reach these or not.
- Linearise the nonlinear system around the equilibrium points and hence study the behaviour of the system trajectories near the equilibrium point using the linearised model.
- Based on the nature of the state trajectories, the equilibrium points can be classified as Node, Focus, Saddle Point and Centre.

Analysis of Singular Points

- For an autonomous, $\dot{x} = f(x)$
- The linearized model is $\delta\dot{x}(t) = \left.\frac{\partial f}{\partial x}\right|_{x_e} \delta x$ (at equilibrium point x_e)

Where $\frac{\partial f}{\partial x}$ is called the Jacobian matrix evaluated at equilibrium or singular points.

- The linearization will result to a linear time invariant system with a constant matrix **A** whose eigen values can be analysed to study the state trajectories.
- The eigen values of the matrix **A** can be both positive, both negative or one positive other negative, complex with negative or positive real part or imaginary.
- Based on these the state trajectories will be different.

Analysis of Singular Points

- Let the linearized system be $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$

i.e. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- The eigen values of matrix $\mathbf{A}$ can be found out from the characteristics equation $|\lambda\mathbf{I} - \mathbf{A}| = 0$

$$\left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} \right| = 0$$

$$(\lambda - a)(\lambda - d) - cb = 0$$

$$\lambda^2 - (a + b)\lambda + ad - bc = 0$$

The eigen values are $\lambda_1 \, and \, \lambda_2$

Analysis of Singular Points

The eigen values are:

$$\lambda_1, \lambda_2 = \left(\frac{a+d}{2}\right) \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - (ad - bc)}$$

Using a linear transformation $\mathbf{x} = \mathbf{Mz}$ the equation can be transformed to canonical form:

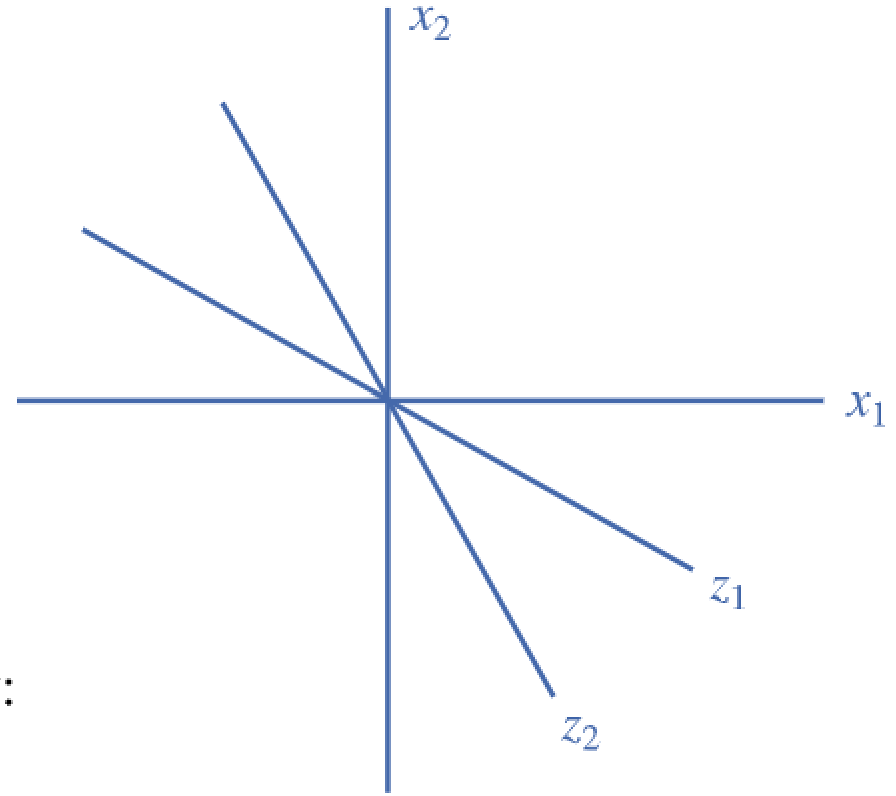
$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- The transformation given simply transforms the coordinate axes from x_1-x_2 plane to z_1-z_2 plane having the same origin, but does not affect the nature of the roots of the characteristic equation.
- The phase trajectories obtained by using this transformed state equation still carry the same information except that the trajectories may be skewed or stretched along the coordinate axes.

Analysis of Singular Points

- In general, the new coordinate axes will not be rectangular as shown in figure.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$



The solution to the state equation can be written as:

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2 z_2$$

With the solution of the two first order equations being given by:

$$z_1(t) = e^{\lambda_1 t} z_1(0)$$

$$z_2(t) = e^{\lambda_2 t} z_2(0)$$

Analysis of Singular Points

The slope of the trajectory in the z_1-z_2 plane is given by:

$$\frac{dz_2}{dz_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1} = \tan\theta$$

$$\frac{dz_2}{z_2} = \left(\frac{\lambda_2}{\lambda_1}\right)\left(\frac{dz_1}{z_1}\right)$$

Upon integration, this gives

$$\ln(z_2) = \left(\frac{\lambda_2}{\lambda_1}\right)\ln(z_1) \quad \text{or} \quad z_2 = c(z_1)^{\lambda_2/\lambda_1}$$

Based on the nature of these eigen values and the trajectory in z_1-z_2 plane, the singular points are classified as follows.

Classification of Singular Points-Node (Stable)

(a) Eigen values are real distinct and negative

If the singular point is at the origin, for real distinct and negative,

$$z_2 = c(z_1)^{\lambda_2/\lambda_1} \qquad z_2 = c(z_1)^{k_1}$$

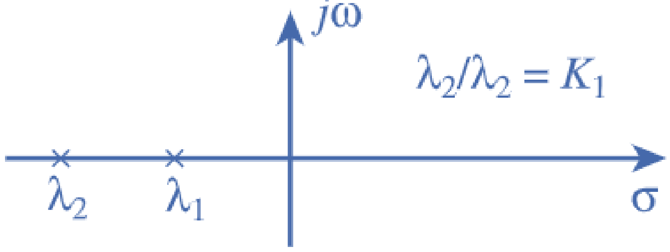
where $k_1 = (\lambda_2/\lambda_1) \geq 0$ so that the trajectories become a set of parabola as shown in figure.

The solution to the state equations can be shown to be:

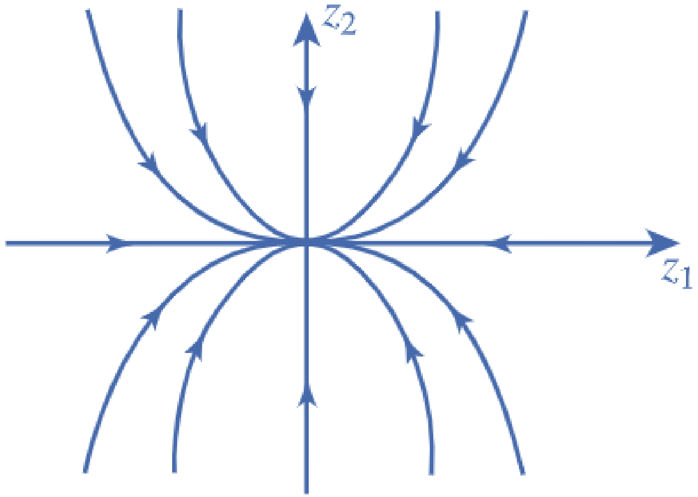
$$z_1(t) = e^{\lambda_1 t} z_1(0) \text{ and } z_2(t) = e^{\lambda_2 t} z_2(0).$$

- The equilibrium point where the trajectories meet is called Node.

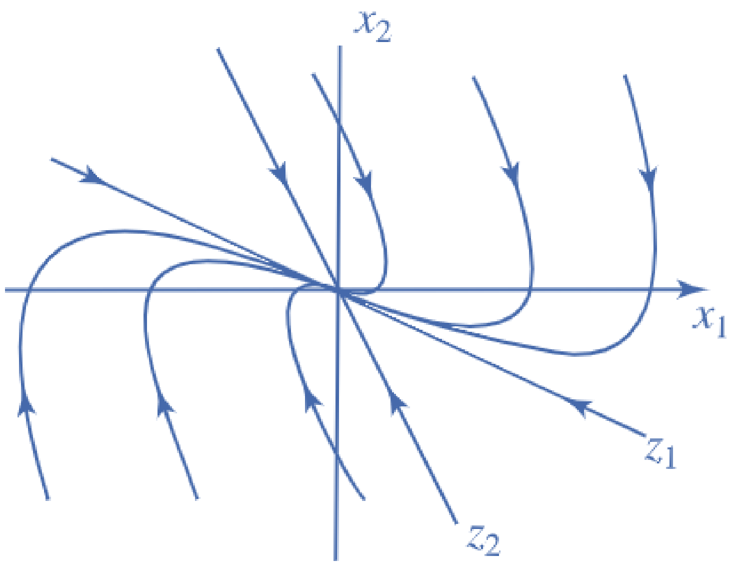
$$z_2 = c(z_1)^{k_1}$$



(a) Location of eigenvalues



(b) Stable node in (z_1, z_2)



(c) Stable node in (x_1, x_2)-plane

Phase trajectories around a stable node

Classification of Singular Points-Node (Unstable)

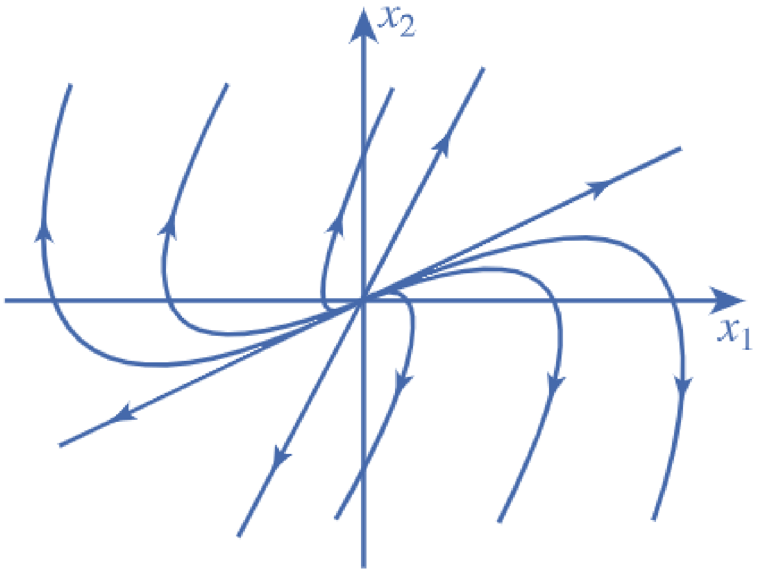
(b) Eigen values are real distinct and positive

If the eigen values are both positive, the nature of the trajectories does not change, except that the trajectories diverge out from the equilibrium point as both $z_1(t)$ and $z_2(t)$ are increasing exponentially.

The solution to the state equations can be shown to be:

$$z_1(t) = e^{\lambda_1 t} z_1(0) \text{ and } z_2(t) = e^{\lambda_2 t} z_2(0).$$

This type of singularity is identified as a node, but it is an *unstable node* as the trajectories diverge from the equilibrium point.



Phase trajectories around unstable node

Classification of Singular Points-Saddle Points

(c) Eigen values are real distinct one positive and other negative

Here, one of the states corresponding to the negative eigen value converges and the one corresponding to positive eigen value diverges so that the trajectories are given by:

$$z_2 = c(z_1)^{-k} \text{ or } (z_1)^k z_2 = c$$

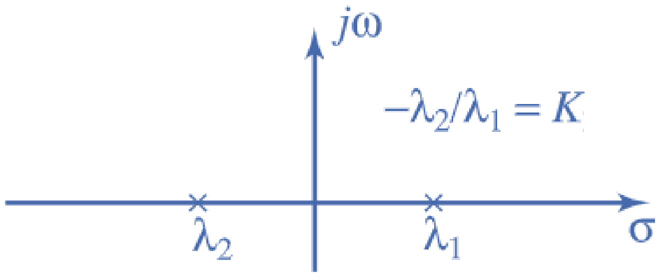
which is an equation to a rectangular hyperbola for positive values of k.

Moreover, the solution to the state equations is of the form

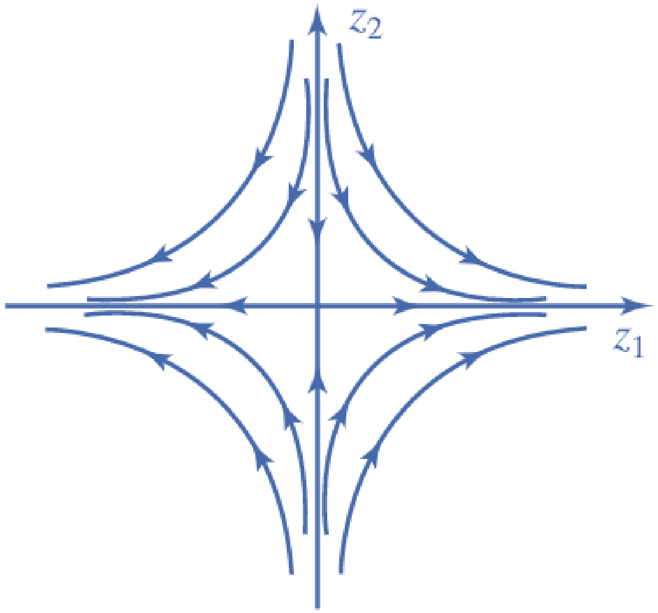
$$z_1(t) = e^{\lambda_1 t} z_1(0) \text{ and } z_2(t) = e^{\lambda_2 t} z_2(0)$$

where one is decreasing and another increasing since λ_1 and λ_2 are of opposite sign.

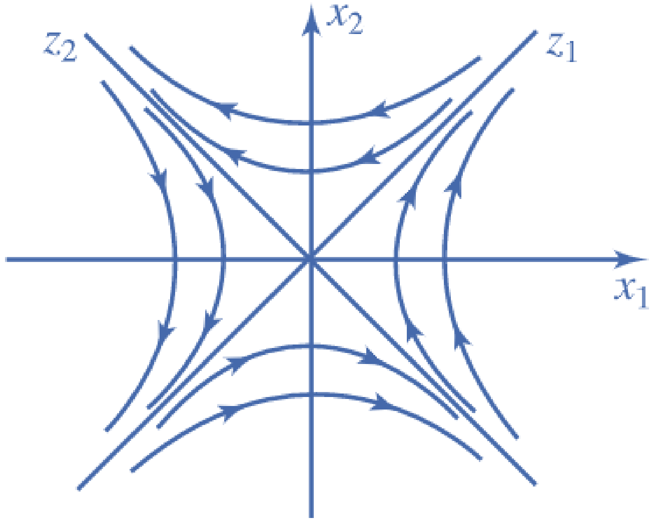
The equilibrium point around which the trajectories are of this type is called a *saddle point*.



(a) Location of eigenvalues



(b) Saddle point in (z_1,z_2)-plane



(c) Saddle point in (x_1, x_2)-plane

Phase trajectories around saddle point

Eigen Values are Complex

Let the eigen values be $\lambda_1, \lambda_2 = \sigma \pm j\omega$. The canonical form of the state equation can be written as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Using the transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1/2 & -j/2 \\ 1/2 & j/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

the new state equation can be shown to be:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\dot{y}_1 = \sigma y_1 + \omega y_2$$

$$\dot{y}_2 = -\omega y_1 + \sigma y_2$$

Eigen Values are Complex

The slope

$$\frac{dy_2}{dy_1} = \frac{-\omega y_1 + \sigma y_2}{\sigma y_1 + \omega y_2} = \frac{y_2 - k y_1}{y_1 + k y_2} \quad \text{where} \quad k = \frac{\omega}{\sigma} \qquad (1)$$

Define $\frac{dy_2}{dy_1} = \tan\psi$ and $\frac{y_2}{y_1} = \tan\theta$

Substituting in Eq. (1)

$$\tan\psi = \frac{\tan\theta - k}{1 + k\tan\theta} \quad \text{or} \quad \tan(\theta - \psi) = k$$

Equation is an equation to a spiral in the polar coordinates.

Classification of Singular Points- Focus (Stable)

(d) Eigen values are complex with negative real part

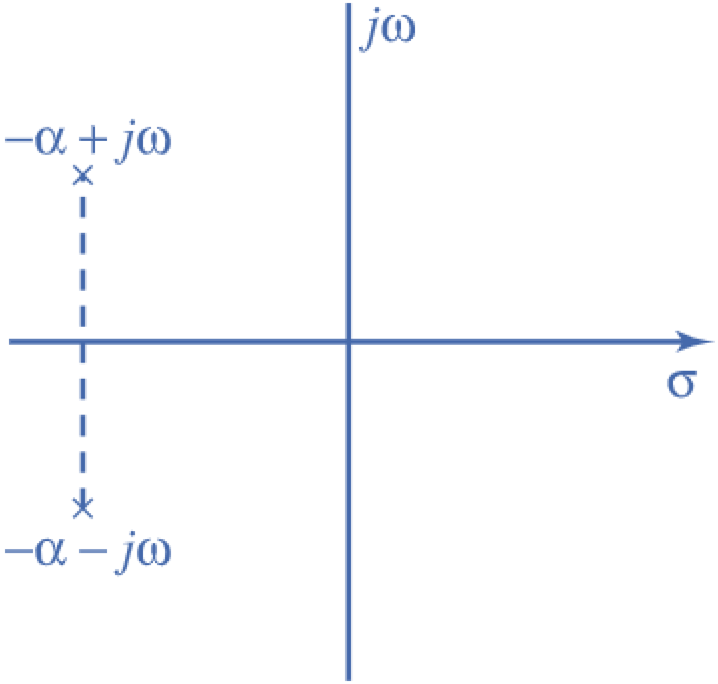
The state responses are both decreasing with respect to time as both are decreasing amplitude sinusoidal signal with:

$$y_1 = c_1 e^{\sigma t} \sin(\omega t) \quad \text{and} \quad y_2 = c_2 e^{\sigma t} \cos(\omega t + \phi)$$

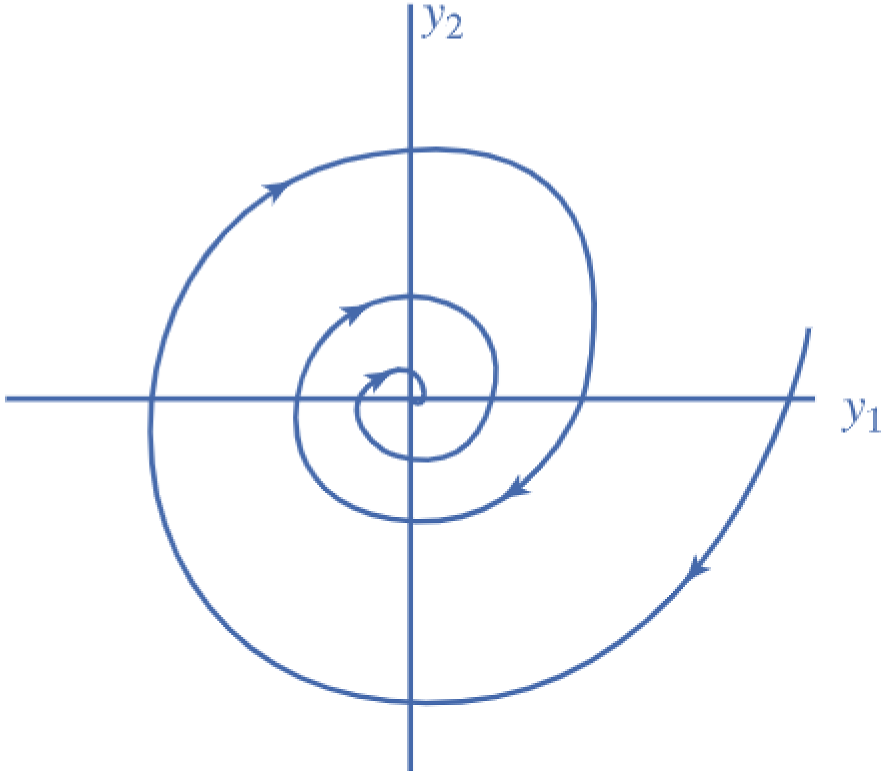
If the real part is negative, the trajectory is spiraling inwards and the equilibrium point to which the trajectory spirals in is called a *stable focus*.

Classification of Singular Points- Focus (Stable)

(d) Eigen values are complex with negative real part



(a) Eigen values



(b) Phase trajectories-stable focus

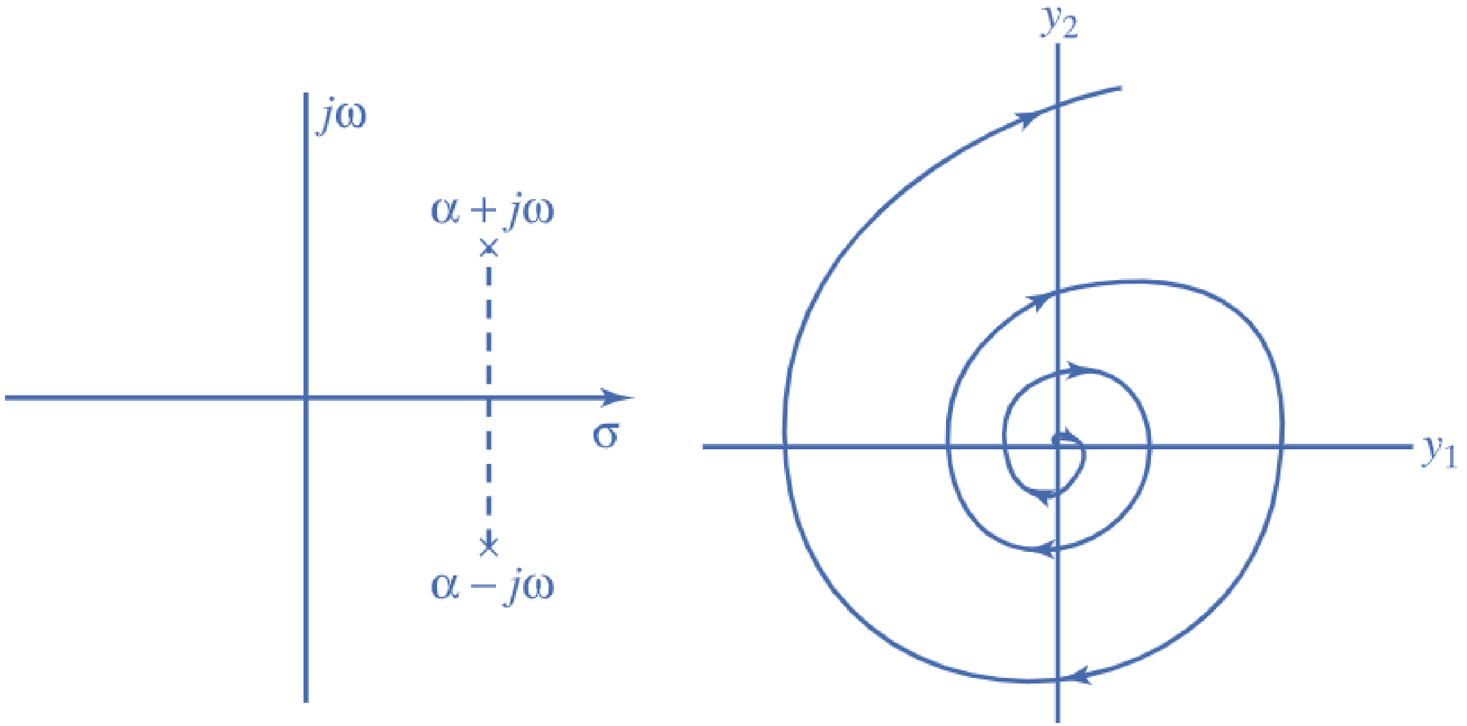
Phase trajectories around stable focus

Classification of Singular Points- Focus (Unstable)

(e) Eigen values are complex with positive real part

The state response is both increasing without bound with respect to time and the trajectories are spiraling out from the equilibrium point.

The equilibrium points are called unstable focus



(a) Eigen values

(b) Phase trajectories unstable focus

Phase trajectories around unstable focus

Classification of Singular Points- Center or Vortex

(f) Eigen values are complex with zero real part

$$\lambda_1, \lambda_2 = \pm j\omega$$

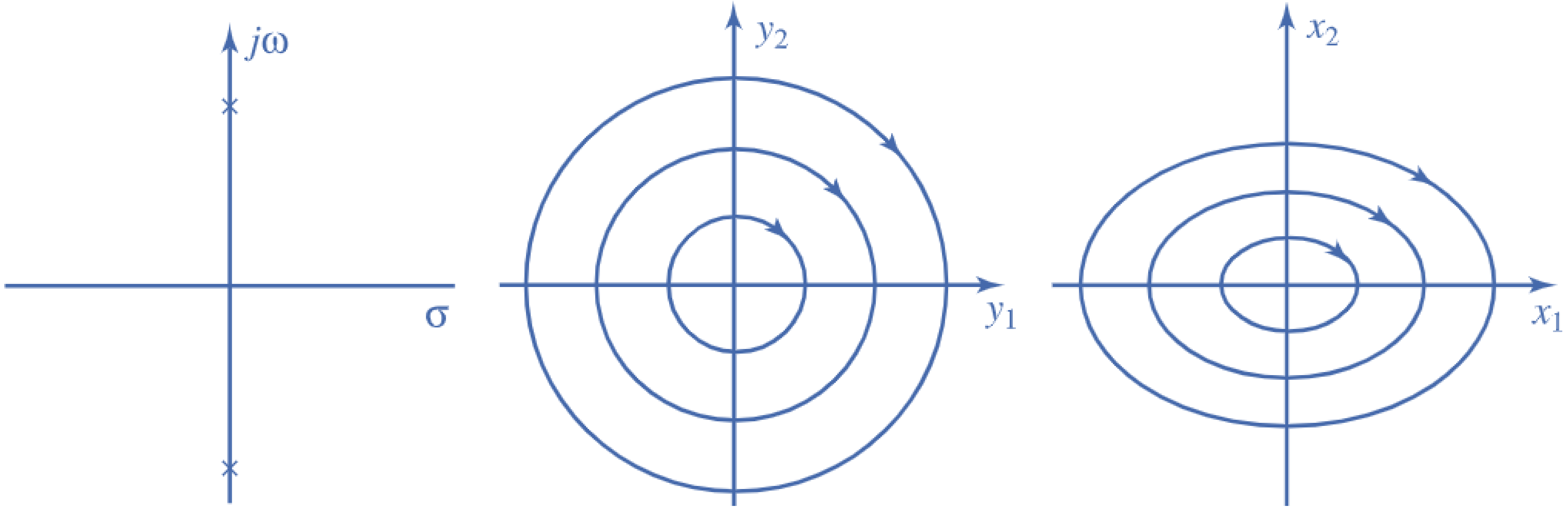
$$\frac{dy_2}{dy_1} = \frac{j\omega y_1}{-j\omega y_2} = -\frac{y_1}{y_2} \text{ from which } y_1 dy_1 + y_2 dy_2 = 0$$

Integrating we get $y_1^2 + y_2^2 = R^2$ which is an equation to a circle of radius R. The radius R can be evaluated from the initial conditions. The trajectories are thus concentric circles in y_1-y_2 plane and ellipses in the x_1-x_2

Such an equilibrium point around which the state trajectories are concentric circles or ellipses is called a *centre* or *vortex*.

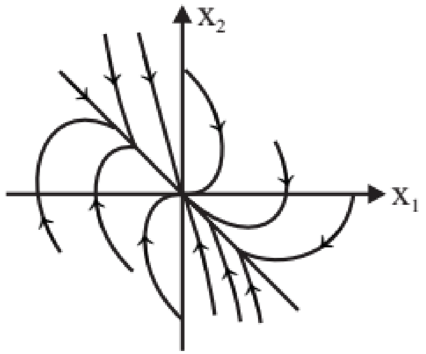
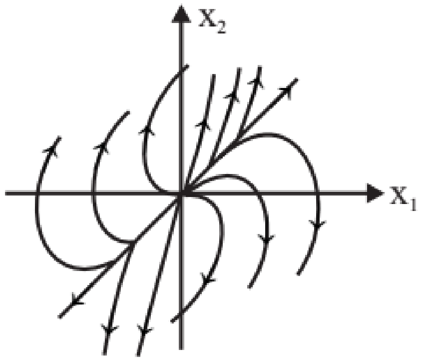
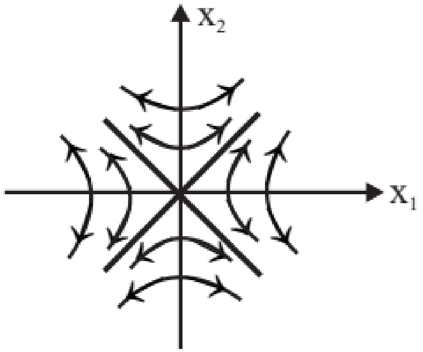
Classification of Singular Points- Center or Vortex

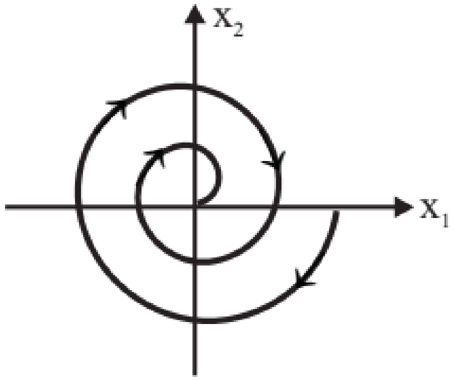
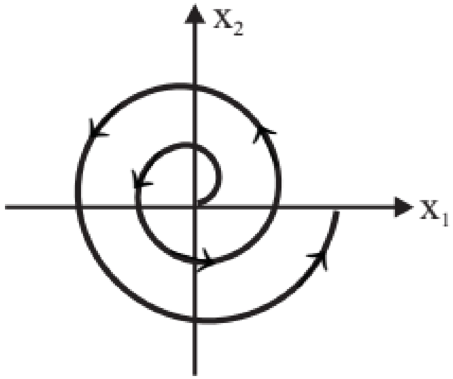
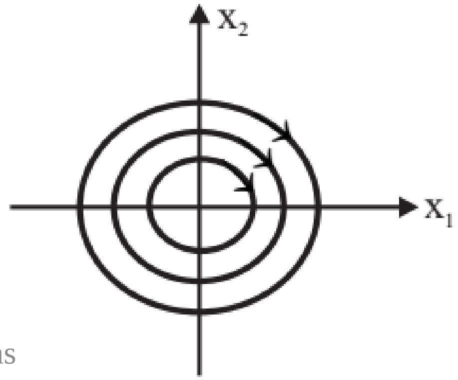
(f) Eigen values are complex with zero real part



(a) Eigen values (b) Center in y_1 - y_2 plane (c) Center in x_1 - x_2 plane

Phase trajectories around vortex or center

Eigen Values of system matrix	**Type of singular point**	**Phase portrait of the system with singular point at origin**
Distinct, real and the two eigen values are negative	Stable node	
Distinct, real and the two eigen values are positive	Unstable node	
Distinct, real, one eigen value is positive and the other is negative	Saddle point	

Eigen Values of system matrix	Type of singular point	Phase portrait of the system with singular point at origin
Complex conjugate with negative real part	Stable focus	
Complex conjugate with positive real part	Unstable focus	
Purely imaginary and conjugate	Centre or Vortex point	

Singular Points-Problem

Q. Consider the nonlinear differential equation:

$$\ddot{y} - \left(0.1 - \frac{10}{3}\dot{y}^2\right)\dot{y} + y + y^2 = 0$$

(a) Find all the singularities of the system.
(b) Classify them.
(c) Sketch the phase portrait in the neighborhood of the singularities.

$$\ddot{y} - \left(0.1 - \frac{10}{3}\dot{y}^2\right)\dot{y} + y + y^2 = 0$$

Solution: Define the state variables as $y = x_1$, $\dot{y} = x_2$. Then the state equations are:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3$$

Singular points, $\dot{x}_1 = \dot{x}_2 = 0$

At the singular points $\dot{x}_1 = x_2 = 0$

$$\dot{x}_2 = -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3 = 0$$

so that at the singular points $x_2 = 0$ and $-x_1 - x_1^2 = 0$ i.e., $x_1(x_1 + 1) = 0$. The singularities are thus $(0, 0)$ and $(-1, 0)$.

Linearisation About the Singularities

Let $f_1(x_1, x_2) = \dot{x}_1 \qquad f_2(x_1, x_2) = \dot{x}_2$

$$f_1(x_1, x_2) = x_2 \qquad f_2(x_1, x_2) = -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3$$

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_2}{\partial x_1} = -1 - 2x_1 \qquad \frac{\partial f_2}{\partial x_2} = 0.1 - 10x_2^2$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

Jacobean matrix is

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -(1 + 2x_1) & 0.1 - 10x_2^2 \end{bmatrix}$$

Linearisation around $(0, 0)$, substituting for $x_1 = 0$ and $x_2 = 0$

$$\mathbf{J}(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0.1 \end{bmatrix} = \mathbf{A}$$

- The eigen values of matrix $\mathbf{A}$ can be found out from the characteristics equation $|\lambda \mathbf{I} - \mathbf{A}| = 0$

Eigen values are roots of: $\lambda(\lambda - 0.1) + 1 = 0$

$$\lambda^2 - 0.1\lambda + 1 = 0$$

$$\lambda_1, \lambda_2 = \frac{1}{2}[0.1 \pm \sqrt{(0.1)^2 - 4}] \quad \lambda_1, \lambda_2 = \frac{1}{2}[0.1 \pm j\sqrt{3.99}]$$

The eigen values are complex with positive real part. The singular point is an unstable focus. The trajectories are diverging spirals from the origin

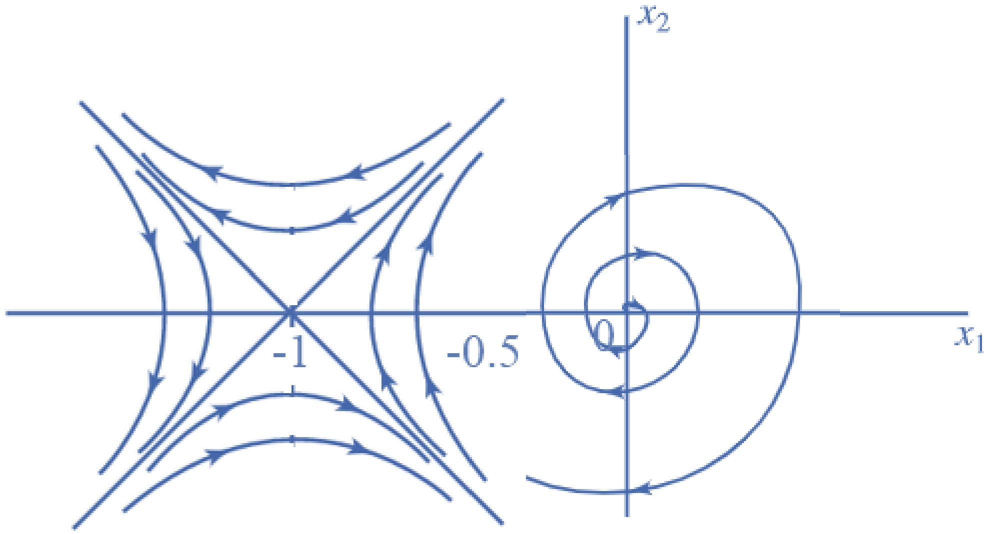
Linearisation around $(-1, 0)$

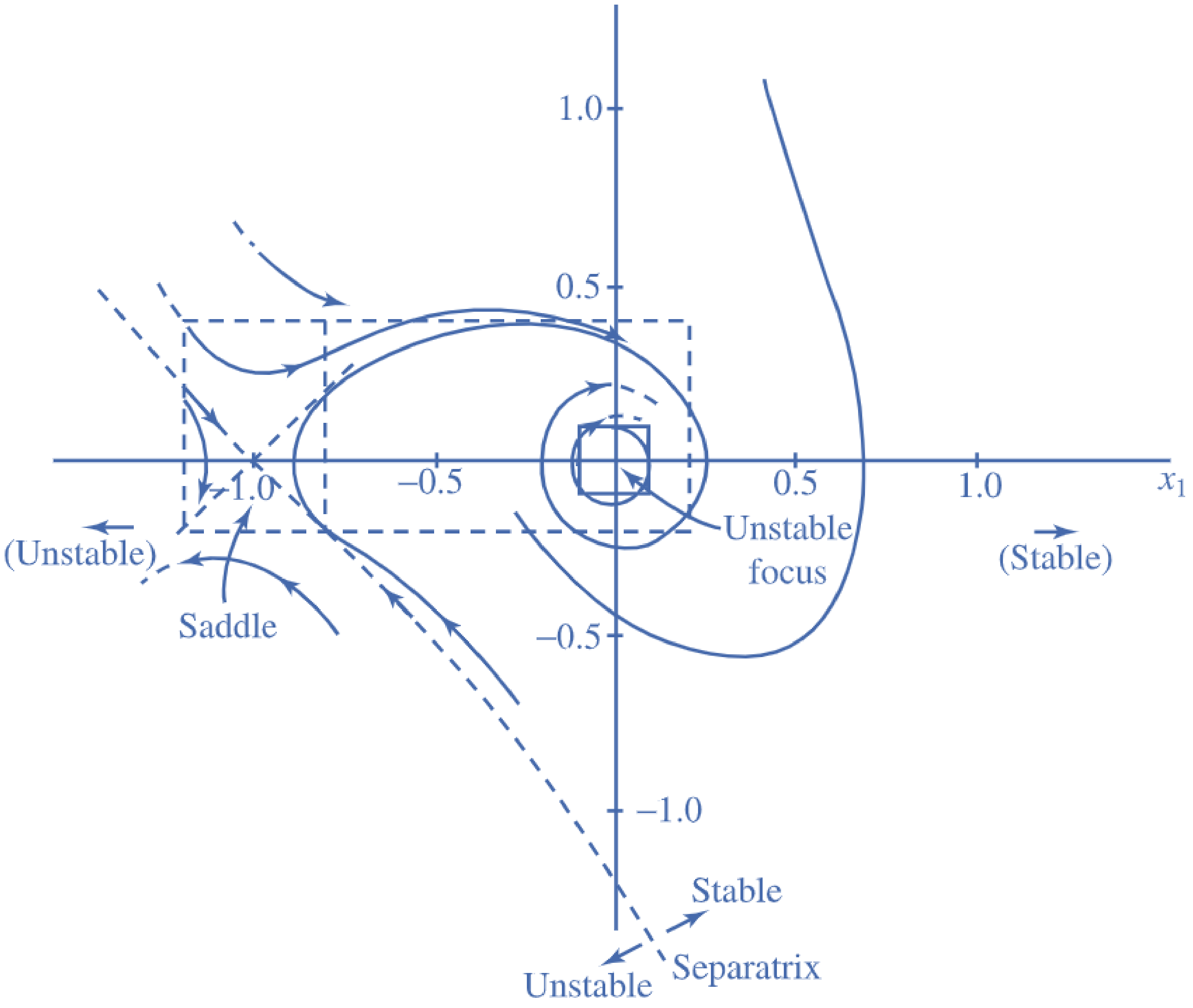
$$\mathbf{J}(-1, 0) = \begin{bmatrix} 0 & 1 \\ -(1-2) & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0.1 \end{bmatrix}$$

Eigen values are roots of $\lambda(\lambda - 0.1) - 1 = 0$ i.e., $\lambda^2 - 0.1\lambda - 1 = 0$

$$\lambda_1, \lambda_2 = \frac{1}{2}[0.1 \pm \sqrt{(0.1)^2 + 4 \times 1}] = 1.05 \text{ and } -0.98$$

Both are real with one negative and another positive. The singular point is a saddle point





Phase portrait-

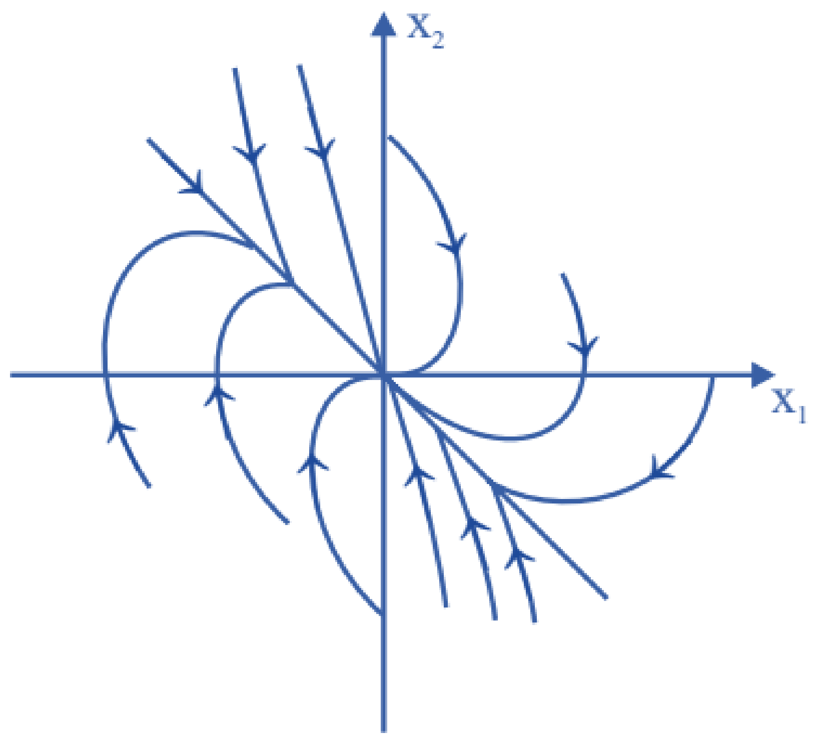
Phase Plane Analysis

Phase Plane Analysis

- The phase plane method of analysis is a graphical method for the analysis of linear and nonlinear systems.
- The analysis is carried by constructing phase trajectories. It gives an idea about the transient behaviour and stability of the system.
- The phase plane analysis is usually restricted to second order systems excited by step or ramp inputs.
- This analysis technique can be extended to a higher order system if it is approximated as a second order system.
- For linear systems the state equations are a set of first order linear differential equations and solutions of state equations can be easily obtained by integration.
- But for nonlinear systems, the state equations are a set of first-order nonlinear differential equations and solving the nonlinear differential equations will not be an easy task.
- Hence for nonlinear systems the phase plane method of analysis will be an useful tool.

Phase Plane and Phase Trajectories

- The coordinate plane with the state variables x_1 and x_2 as two axes is called the phase plane. (i.e., in phase plane, x_1 is represented in x-axis and x_2 in y-axis).
- The curve describing the state point (x_1, x_2) in the phase-plane with time as running parameter is called phase trajectory
- i.e., the locus of state point (x_1, x_2) in phase plane is called phase trajectory.
- A trajectory can be constructed in phase-plane for each set of initial conditions.
- Hence a family of trajectories can be constructed for a system in a phase plane and such a family of trajectories is called a phase portrait.



Singular Points

- A point in the phase-plane at which the derivative of all state variables are zero is called singular point. It is also called an equilibrium point.
- If the system is placed at a singular or equilibrium point, it will continue to lie there if left undisturbed i.e., the derivatives of all the phase variables are zero and so the system state remains unchanged.
- The state equations of a system are formed by taking the first derivatives of state variables as functions of state variables and inputs.
- If the inputs are zero or constants then the state equation will be in the form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$
- A system represented by equation is called an autonomous system. The linearized model of the system represented by equation may be written as

$$\dot{X} = AX$$

Let, X_e = States corresponding to singular point or equilibrium state.

At equilibrium state, $\dot{X} = 0$

$$\therefore \; A X_e = 0$$

Since the determinant of A is non-zero, $X_e = 0$, will be the only solution for equation

- Therefore it can conclude that if all the eigen values of the system are non-zero then the origin is the only singular point.

Stability Analysis of Nonlinear Systems using Phase Trajectories

- For linear time invariant systems, the concept of stability can be defined as follows,
  1. When the input is zero, the system is stable for arbitrary initial conditions if the resulting trajectory tends towards the equilibrium state.
  2. When the system is excited by a bounded input, the system is stable if the system output is bounded.
- In nonlinear systems the concept of stability is not clear-cut. There are many types of stability definitions in the literature.
- The linear autonomous systems has only one equilibrium state. The behaviour of linear system about the equilibrium state completely determines the qualitative behaviour in the entire state-plane.
- In nonlinear systems there may be multiple equilibrium state. The behaviour of nonlinear system about the equilibrium point may be different for small deviations and large deviations about the equilibrium point.
- In nonlinear systems with multiple equilibrium states, the system trajectories may move away from one equilibrium point and tend to other as time progresses.
- Hence in nonlinear systems, stability is discussed relative to the equilibrium state and the general stability of a system cannot be defined.

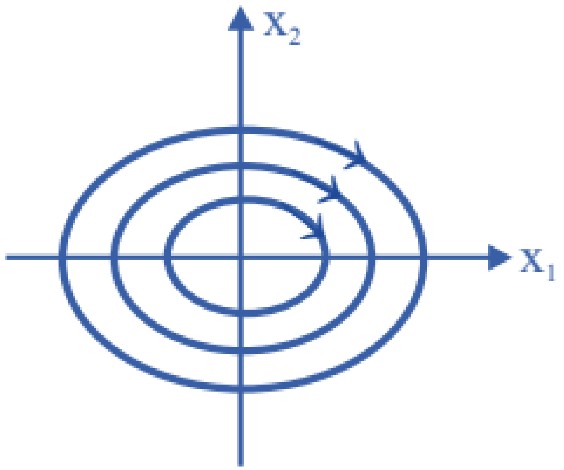
Stability Analysis of Autonomous System

- Consider an autonomous system described by the state equation,
- Let us assume that the system has one equilibrium point and the origin of phase plane is the equilibrium point.
- For this system, the following definitions of stability are proposed.

1. The autonomous system defined by equation $\dot{x} = F(x)$ is stable at the origin, if for every initial state $x(t_0)$ which is sufficiently close to origin, $x(t)$ remains near the origin for all t.
2. The autonomous system defined by equation $\dot{x} = F(x)$ is asymptotically stable if $x(t)$ approaches the origin as $t \rightarrow \infty$.
3. The autonomous system defined by equation $\dot{x} = F(x)$ is asymptotically stable in the large if it is asymptotically stable for every initial state regardless of how near or far it is from the origin.

Limit Cycles in Phase-portrait- Linear System

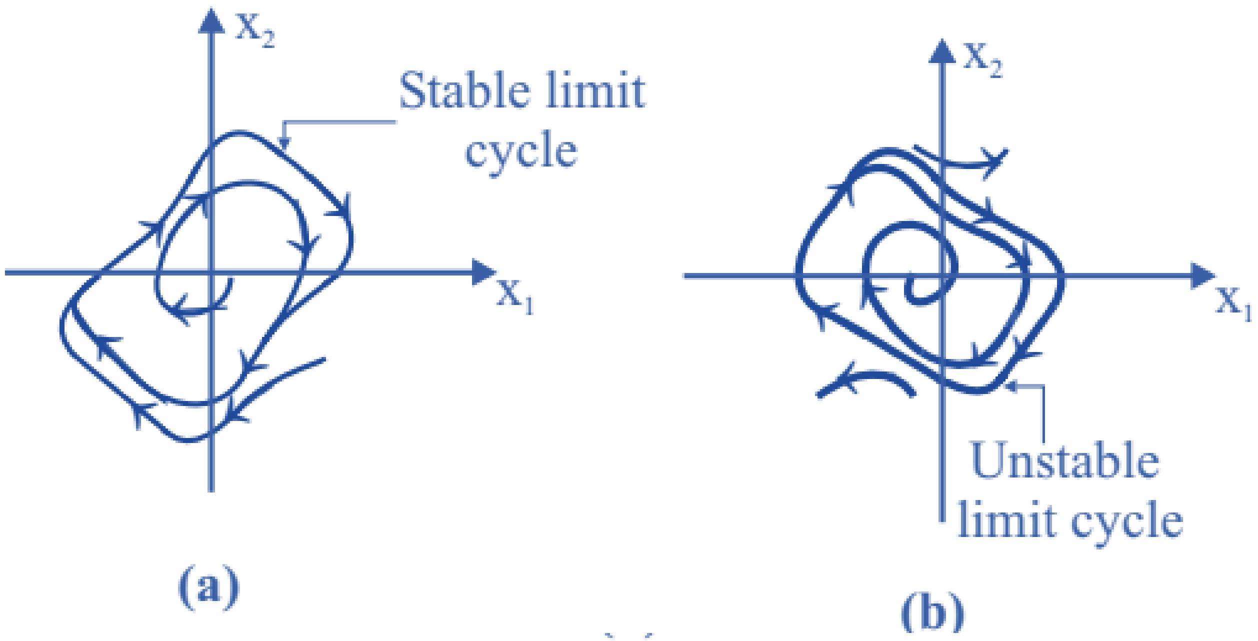
- The limit cycles are oscillations of fixed amplitude and period.
- The existence of limit cycles in nonlinear systems can be predicted from closed trajectories in the phase-portrait.
- In linear systems, when oscillations occur, the resulting trajectories will be closed curves as shown in figure.
- The amplitude of the oscillations is not fixed. It changes with the size of the initial conditions. Slight changes in system parameters will destroy the oscillations.



Phase-portrait showing limit cycle behaviour in linear system.

Limit Cycles in Phase-portrait-Nonlinear System

- In nonlinear systems, there can be limit cycles (oscillations) that are independent of the size of initial conditions.
- These limit cycles are usually less sensitive to system parameter variations.
- Limit cycles of fixed amplitude and period can be sustained over a finite range of system parameters.
- The limit cycle is stable if the paths in its neighbourhood converge towards the limit cycle figure (a).
- The limit cycle is unstable, if the paths in the neighbourhood of a limit cycle diverge away from it figure (b).



Phase-portrait showing limit cycle behaviour in nonlinear system.

Construction of Phase Trajectories

- The state equation of the second order autonomous system is

$$\dot{x}_1 = f_1(x_1, x_2) = \frac{dx_1}{dt} \text{------}(1)$$

$$\dot{x}_2 = f_2(x_1, x_2) = \frac{dx_2}{dt} \text{--------}(2)$$

Where x_1 and x_2 are the state variables.

Now [(2) ÷ (1)]

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}$$

- This equation defines the slope of the trajectory at every point in the phase plane except at singular point where slope is indeterminate.

- The phase trajectory is constructed using this slope equation analytically and graphically (isocline method).

Construction of Phase Trajectories- Isocline method

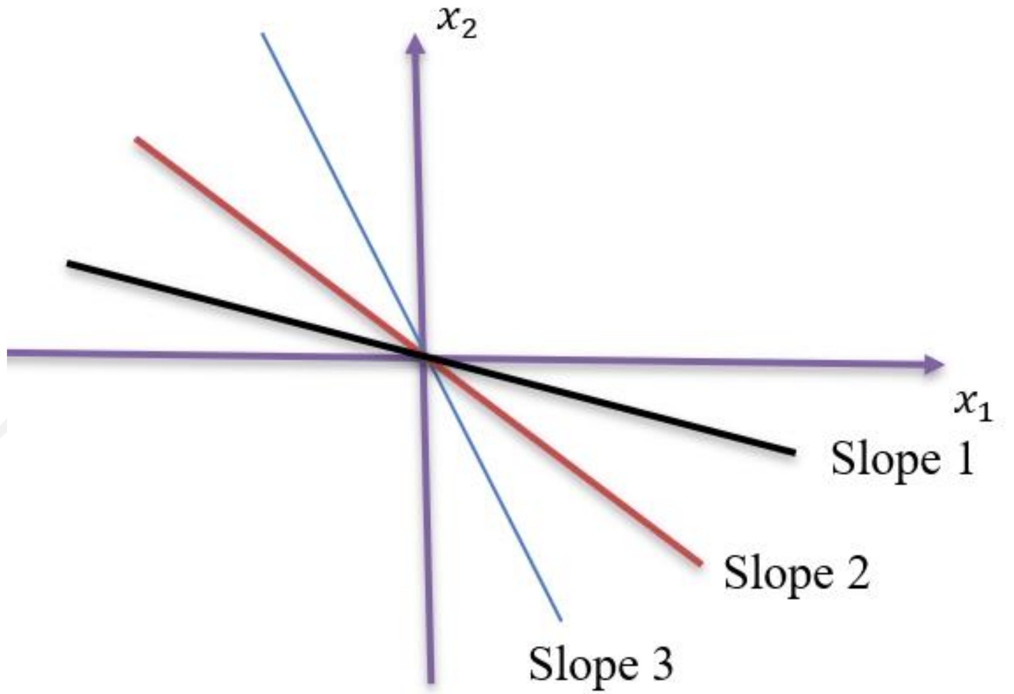
- Let be the **s** slope of a point on phase trajectory in phase plane

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)} = s$$

- For a specific trajectory slope of s_n

$$f_2(x_1,x_2) = s_n\, f_1(x_1,x_2)$$

- The above equation represent many points in phase plane for which slope is equal to s_n.
- A locus passing through the points of same slope in phase plane is called ***isocline***.
- The slope of a phase trajectory at the crossing point of an isocline will be the slope of corresponding isocline.



Construction of Phase Trajectories- Isocline method -Procedure

1. Write the system equation in state phase variable form.

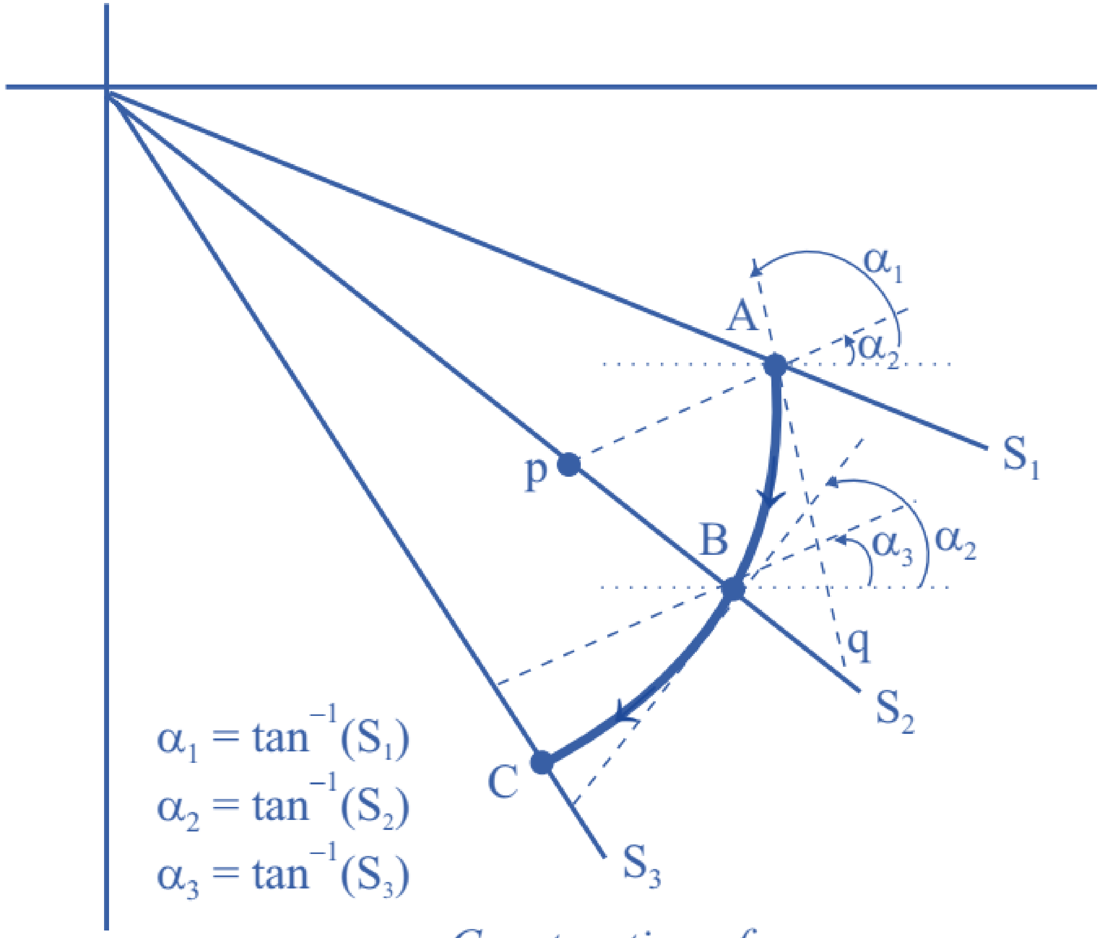
2. Determine the equation of the slope $\frac{\dot{x}_2}{\dot{x}_1} = \frac{dx_2}{dx_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}$

3. By substituting various numeric values of s draw the isoclines $s_1\ s_2 \ldots s_n$

4. Locate the initial points using the given initial conditions.
   - Let point A on isocline-1 be the point corresponding to a set of initial conditions.
   - The phase trajectory will leave the point A at a slope of s_1.
   - When the trajectory reaches isocline-2 the slope changes to $s_{2.}$

Construction of Phase Trajectories- Isocline method -Procedure

5. Draw two lines from point A one at slope s_1 and the other at slope s_2.
   - These lines meet isocline-2 at p and q.
   - Trajectory would cross the isocline-2 at a point B which is midway between p and q
6. Follow the same step to find C, D,…
7. Draw a smooth curve joining A, B, C, D… points which constitute the phase portrait of the system.



Construction of phase trajectory by isocline method.

Isocline method -Problem

Q. A linear second order servo is described by the equation

$$\ddot{e} + 2\zeta\omega_n \dot{e} + \omega_n^2 e = 0$$

where $\zeta = 0.15$, $\omega_n = 1$ rad/sec, $e(0) = 1.5$ and $\dot{e}(0) = 0$

Determine the singular point. Construct the phase trajectory, using the method of isoclines.

SOLUTION

STEP 1: Let x_1 and x_2 be the state variables of the system and they are related to the system variable, e as shown below.

$$x_1 = e$$

$$x_2 = \dot{e}$$

On differentiating equation

$$\dot{x}_1 = \dot{e}_1 = x_2$$

On differentiating equation

$$\dot{x}_2 = \ddot{e}$$

Given that, $\ddot{e} + 2\zeta\omega_n \dot{e} + \omega_n^2 e = 0$

$$\dot{x}_2 + 2\zeta\omega_n x_2 + \omega_n^2 x_1 = 0$$

$$\therefore \dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2$$

The state equations of the system are given by equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2$$

The singular point is obtained from state equations by putting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$

Let the coordinates of singular point in phase plane = (x_1^0, x_2^0)

$$\dot{x}_1 = x_2 \quad \dot{x}_1 = 0 \text{ and } x_2 = x_2^0 \quad x_2^0 = 0.$$

$$\dot{x}_2 = -\omega_n^2 x_1 - 2\zeta\omega_n x_2$$

$$0 = -\omega_n^2 x_1^0 - 2\zeta\omega_n^2 x_2^0$$

But $x_2^0 = 0, \quad \therefore \; 0 = -\omega_n^2 x_1^0 \quad \text{(or)} \quad x_1^0 = 0$

Therefore, the coordinates of singular point are (0, 0) and so the origin is the singular point.

STEP 2: The slope of the phase trajectory is given by

$$S = \frac{dx_2/dt}{dx_1/dt} = \frac{\dot{x}_2}{\dot{x}_1}$$

On substituting for $\dot{x}_1$ and $\dot{x}_2$

$$S = -\frac{(\omega_n^2 x_1 + 2\zeta\omega_n x_2)}{x_2}$$

Put $\zeta = 0.15$ and $\omega_n = 1$, in equation

$$S = -\frac{(x_1 + 2 \times 0.15 x_2)}{x_2} = \frac{-(x_1 + 0.3x_2)}{x_2}$$

$$= \frac{-x_1}{x_2} - \frac{0.3x_2}{x_2} = \frac{-x_1}{x_2} - 0.3$$

$$\therefore \frac{x_1}{x_2} = -0.3 - S \quad \text{(or)} \quad \frac{x_2}{x_1} = \frac{1}{-0.3 - S}$$

STEP 3:Construct the Isoclines

$$\therefore \; x_2 = \frac{x_1}{-0.3 - S}$$

- From the above equation it can be concluded that isoclines are straight lines.
- For each value of S we can draw one isocline.
- Using the above equation the co-ordinate (x_1, x_2) in the phase plan for various slopes can be calculated.
- Since there are three variables. Let us assume two variables and calculate the third variable.

Let us choose values of S as –2, –1.0, –0.5, 0, 0.5, 1.0 and 2.0

For each value of S, choose two values of x_1 and calculate x_2 using equation

- The slope angle, α is calculated for each value of S, using the expression, $\alpha = \tan^{-1}(S)$

S	–2.0		–1.0		–0.5		0		0.5		1.0		2.0	
α	–63°		–45°		–27°		0		27°		45°		63°	
	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
	1.0	0.6	1.0	1.4	0.25	1.25	0.25	–0.8	1.0	–1.25	1.0	–0.77	1.0	–0.43
	2.0	1.2	1.5	2.1	0.5	2.5	0.75	–2.5	2.0	–2.5	2.0	–1.54	2.0	–0.86

The isoclines corresponding to each slope is drawn using the coordinates

STEP 4: Initial Conditions

The phase trajectory starts at point A on x_1 axis, (i.e., given initial condition is (1.5, 0)).

STEP 5:

- The slope of the trajectory when it crosses x_1 axis is infinite.
- Hence draw a line at an angle, $\tan^{-1} \alpha = 90^0$ with respect to x_1 axis and passing through point A.
- Let this line meet the isocline corresponding to S = 2 at point q.
- Then draw a line at an angle $\tan^{-1} 2 = 63^0$ with respect to x_1 axis and passing through point A.
- Let this line meet the isocline corresponding to S = 2 at point p.
- Now the phase trajectory will pass through point B on the isocline corresponding to S_2. The point B lies at the middle of the segment pq.
- Draw a smooth curve between A and B, which is a section of phase trajectory.

STEP 6:

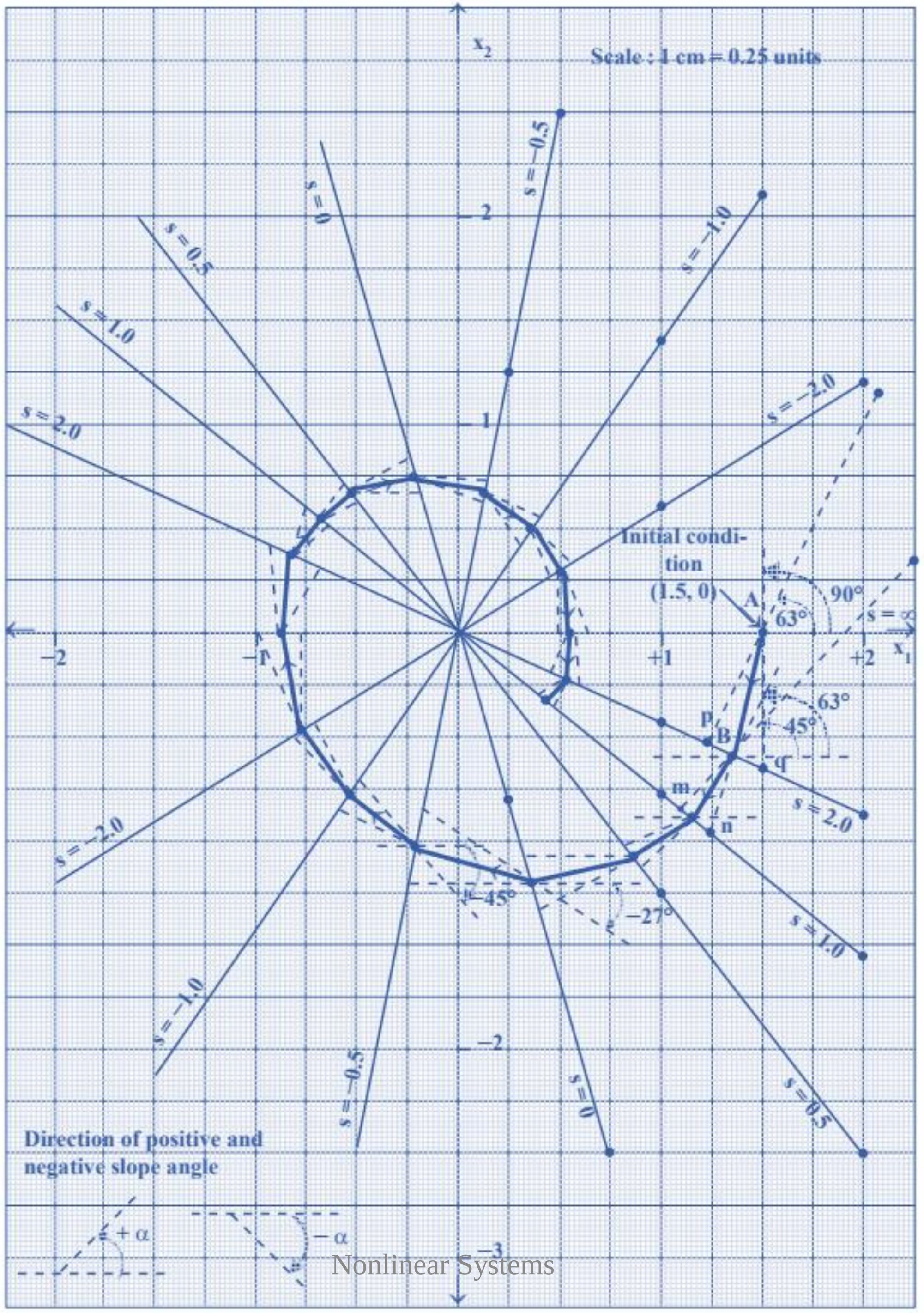
- In general, if F is the crossing point of phase trajectory with isocline-a corresponding to a slope of S_a and if the next isocline is isocline-b corresponding to a slope of S_b.
- Then draw two lines through point F, one at an angle of $\tan^{-1} S_a$ and the other at $\tan^{-1} S_b$.
- The angles are marked from point F with respect to a (horizontal) line parallel to x_1 axis. Positive angles are measured in anticlockwise direction and negative angles are measured in clockwise direction.
- These two lines will meet the isocline-b at points d and e. Now the crossing point of phase trajectory is fixed at the middle of d and e on the isocline-b.

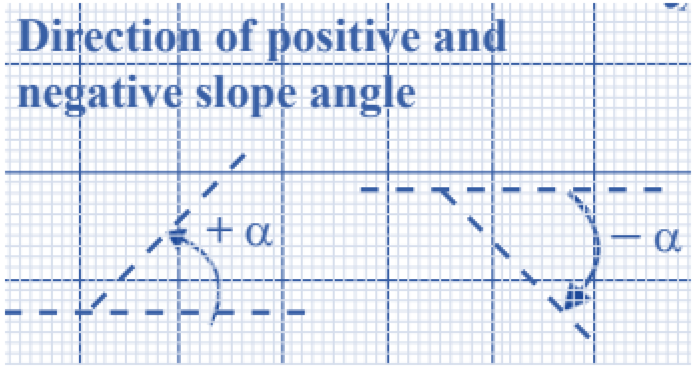
STEP 7:

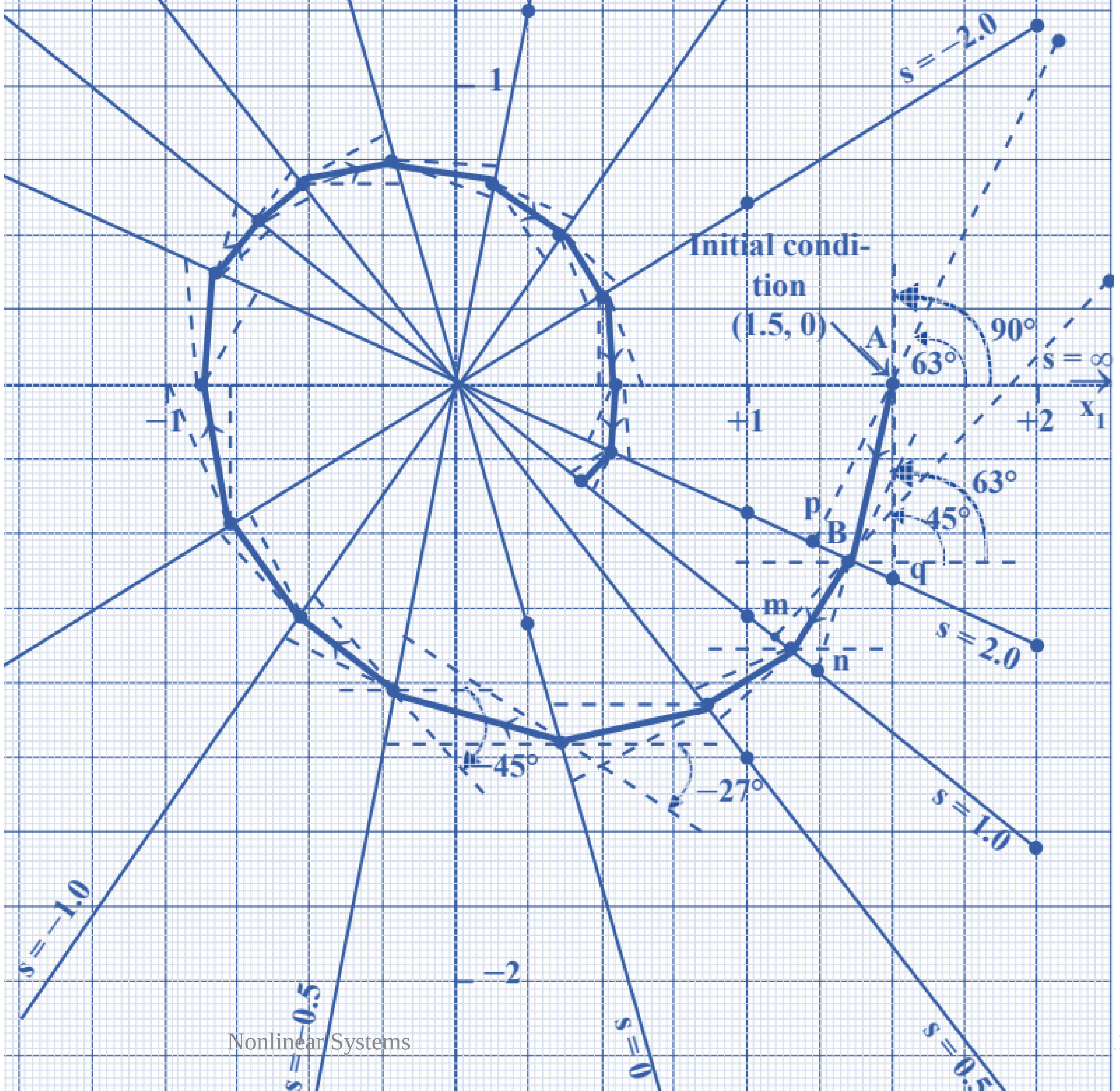
- Thus crossing point of trajectory on each isocline is determined.
- The complete phase trajectory is obtained by drawing a smooth curve through all the crossing points, as shown figure.

Result

- The singular points lies at the origin.
- The phase trajectory spiral towards the origin, hence the type of singular point is stable focus.


x2
Scale : 1 cm = 0.25 units
s = −0.5
s = 0
s = 0.5
s = 1.0
s = 2.0
s = −1.0
s = −2.0
Initial condi-
tion
(1.5, 0)
A
90°
63°
s = ∞
x1
−2
−1
+1
+2
2
1
−2
−3
63°
45°
p
B
q
m
n
−45°
−27°
Direction of positive and
negative slope angle
+ α
− α

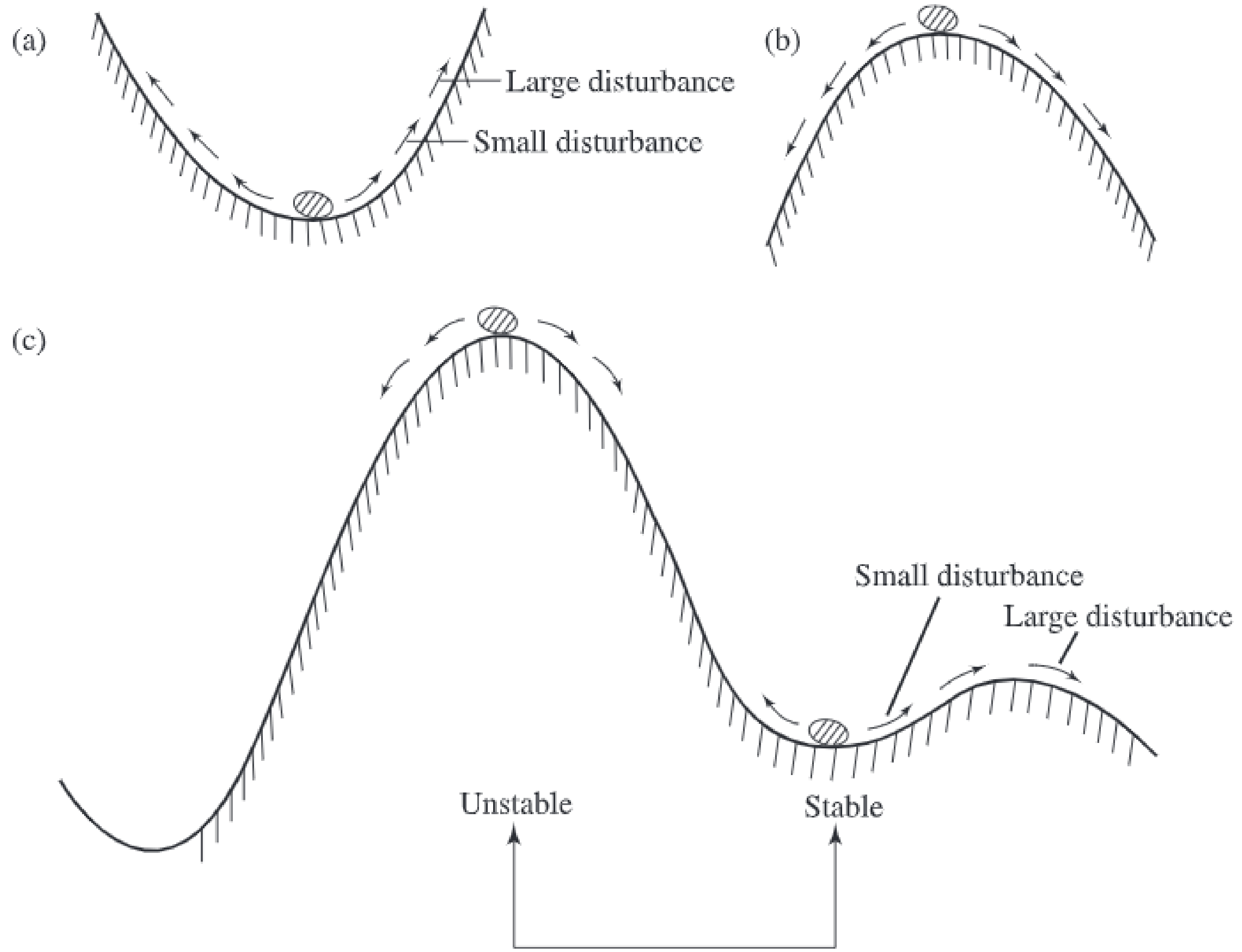

Direction of positive and negative slope angle
+ α
− α


s = −2.0
1
Initial condition (1.5, 0)
A
90°
63°
s = ∞
−1
+1
+2
x_1
63°
p
B
45°
q
m
n
s = 2.0
−45°
−27°
s = 1.0
s = −1.0
−2
s = −0.5
s = 0
s = 0.5

Lyapunov Stability Analysis

Stability in the Small and Stability in the Large

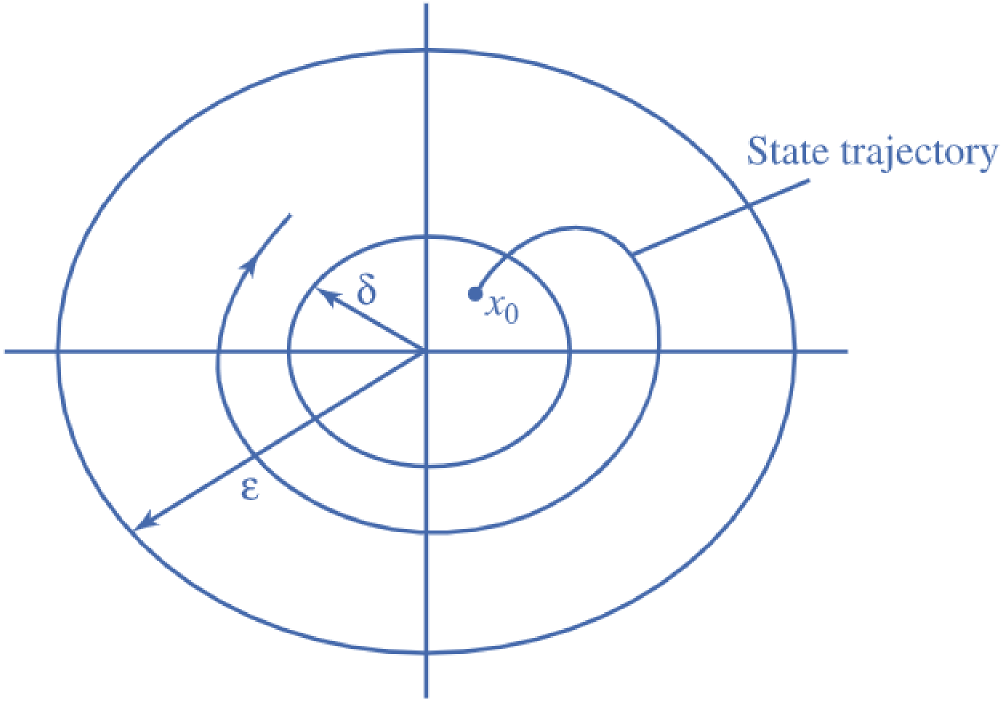
- As far as the the stability of a nonlinear system is concerned, there is no point in talking about system stability.
- More meaningful will be to talk about the stability of an equilibrium point.
- Stability in a region close to the equilibrium point or in the immediate neighbourhood of the equilibrium point is called stability in the small.
- For a larger region around the equilibrium point, the stability may be referred to as stability in the large.
- Stability of a trajectory starting from anywhere in the complete state space can be called as global stability.



(a) Global stability (stable in the large) (b) Global instability (unstable in the large) (c) Local stability

Lyapunov Stability Definition - Stability

Definition 1: Lyapunov stability: An equilibrium state $\mathbf{x}_e$ of an autonomous dynamic system is stable (or stable in the sense of Lyapunov) if for every $\varepsilon > 0$, there exists a $\delta > 0$ where δ depends only on ε, such that $\|\mathbf{x} - \mathbf{x}_e\| < \delta$ results in $\|\mathbf{x}(t_0; x_0) - x_e\| \leq \varepsilon$ for all $t \geq t_0$.



Lyapunov's stability

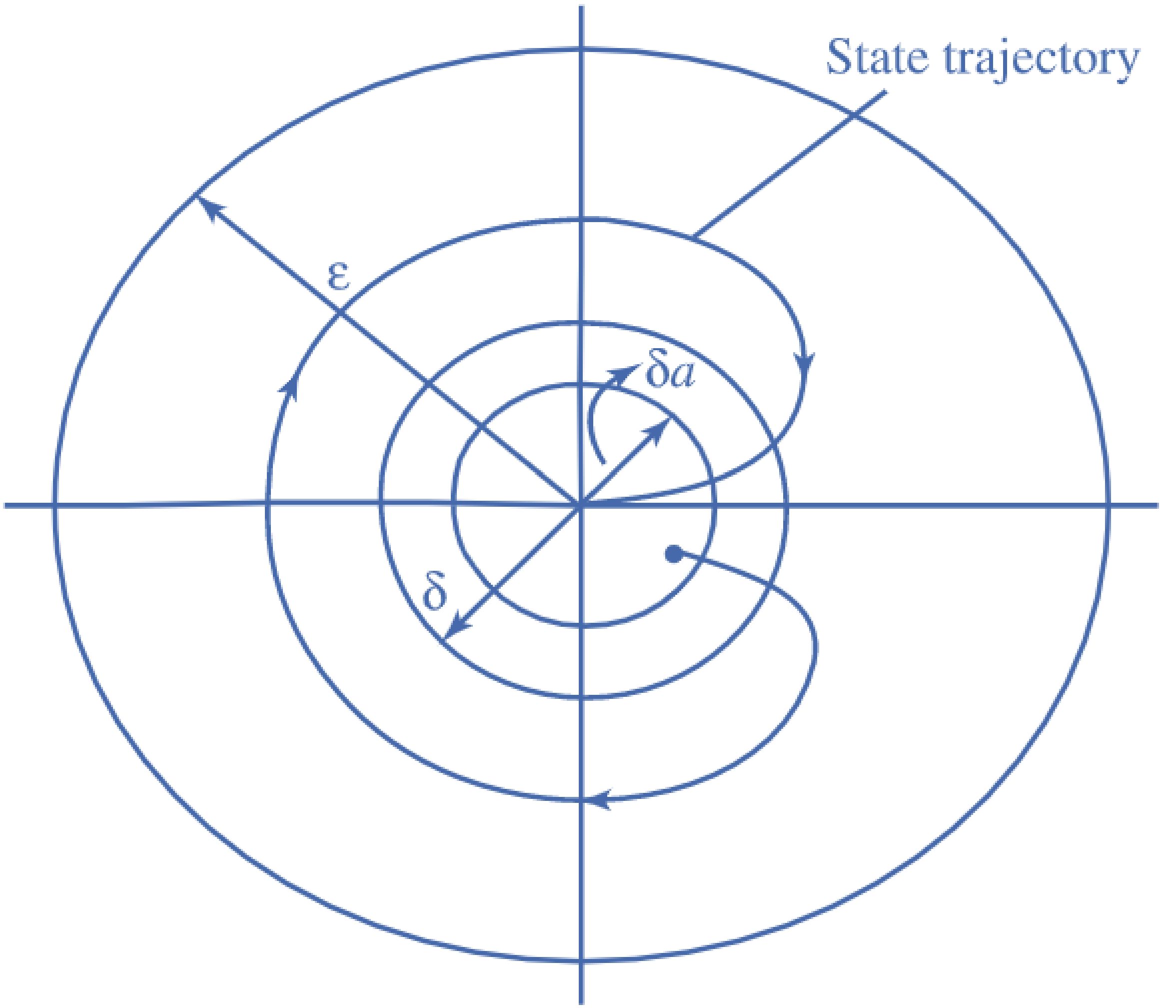
Lyapunov Stability Definition – Asymptotic Stability

Definition 2: Asymptotic stability: An equilibrium state $\mathbf{x}_e$ of an autonomous dynamic system is asymptotically stable if

1. it is stable in the sense of Lyapunov
2. there is a number δ_a such that every motion starting in the neighborhood δ_a of $\mathbf{x}_e$ converges to $\mathbf{x}_e$ as t tends to ∞

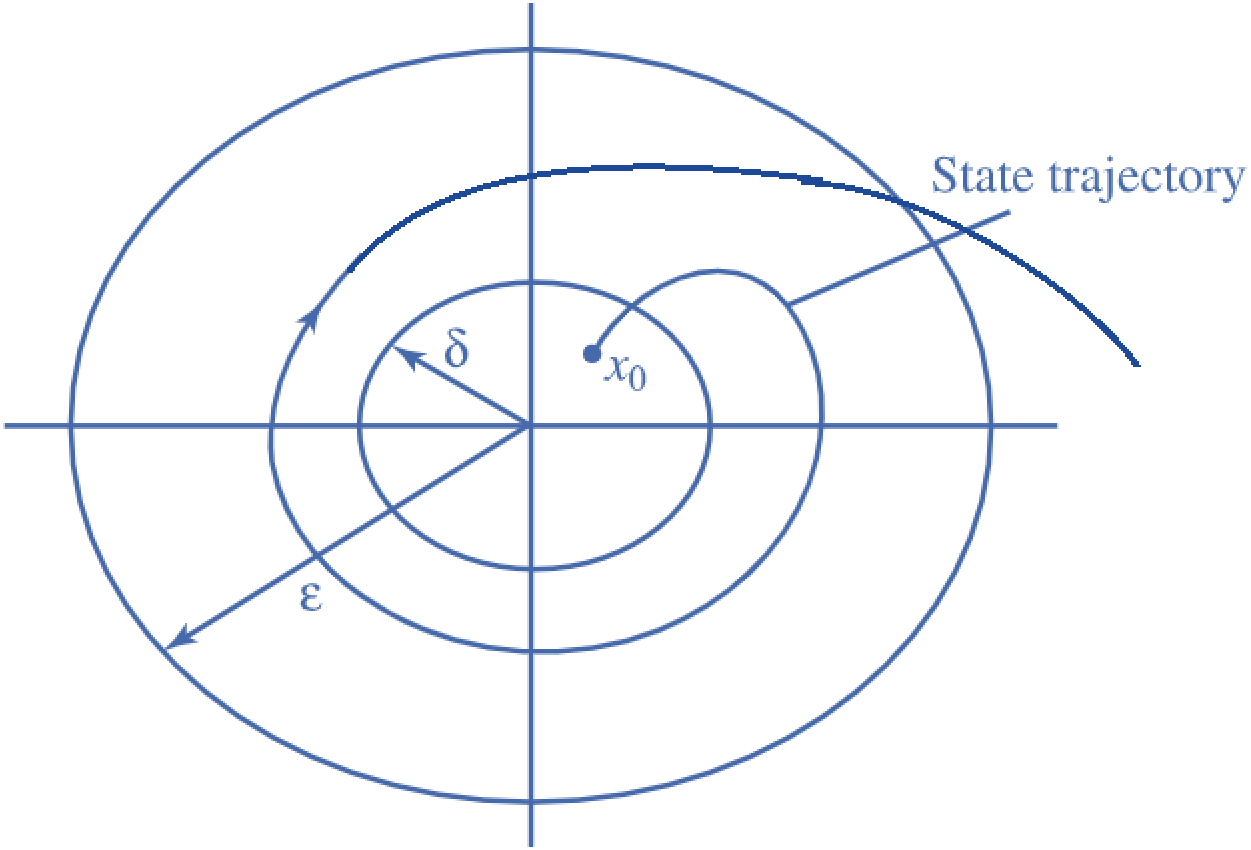
If the above property 2 is observed, but not property 1, the system is said to be *quasi-asymptotically stable*.

Lyapunov Stability Definition – Asymptotic Stability



Lyapunov Stability Definition –Instability

Definition 3: An equilibrium state $\mathbf{x}_e$ of a free dynamic system is unstable if there exists an ε such that no δ can be found to satisfy the conditions of Asymptotic Stability.



Stability by the Method of Lyapunov

- For analysis the stability A. M. Lyapunov suggested two methods
  1. First method or indirect method
  2. Second method or direct method (popular method)

First method or indirect method

- The first method of Lyapunov, though rarely talked about, is essentially a theorem stating the conditions under which system stability information can be inferred by examining the simplified equations obtained through local linearisation.
- This theorem is applicable only to autonomous systems.
- The theorem can be stated as follows:

Stability by the First Method of Lyapunov

- The theorem can be stated as follows:

Theorem: For an autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ let $\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x}_e)}{\partial \mathbf{x}}\delta\mathbf{x} + h(\mathbf{x}_e, \delta\mathbf{x})$ be the equation of the perturbed system about an equilibrium state x_e, if

$$\lim_{\|\delta\mathbf{x}\| \to 0} \frac{h(\mathbf{x}_e, \delta\mathbf{x})}{\|\delta\mathbf{x}\|} = 0$$

then:

(1) if the linearised system $\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x}_e)}{\partial \mathbf{x}}\delta\mathbf{x}$ has only eigen values with negative real parts, $\mathbf{x}_e$ is asymptotically stable;

(2) if the linearised system has one or more eigen values with positive real parts, $\mathbf{x}_e$ is unstable;

(3) If the linearised system has one or more eigen value with zero real parts and the remaining eigen values have negative real parts, the stability of $\mathbf{x}_e$ cannot be ascertained by studying the linearised system alone, even stability 'in the small'.

Sign Definiteness of Scalar Functions

- Let $V(x_1, x_2, \ldots, x_n)$ be a scalar function of the state variables $x_1, x_2, \ldots, x_n$. Then the following definitions are useful for the discussion of Lyapunov's second method.

Definition: Positive definiteness: A scalar function $V(x_1, x_2, \ldots, x_n)$ is said to be positive definite if $V(\mathbf{x})$ is such that $V(\mathbf{x})$ is positive at all points in the state space except at the origin $\mathbf{x} = 0$, where $V(\mathbf{x})$ is equal to zero.

Definition: Positive semi-definiteness: A scalar function $V(x_1, x_2, \ldots, x_n)$ is said to be positive semi-definite if $V(\mathbf{x})$ is such that $V(\mathbf{x})$ is positive at all points in the state space except at one or more points in the state space including the origin $\mathbf{x} = 0$, where it is equal to zero.

Thus the difference between positive definite and positive semi-definite is that a positive definite function will be zero only at the origin whereas the semi-definite function can be zero at several points in the state space. However, the function remains positive at all other places.

Sign Definiteness of Scalar Functions

Definition: Negative definiteness: A scalar function $V(x_1, x_2, \ldots, x_n)$ is said to be negative definite if $V(\mathbf{x})$ is such that $V(\mathbf{x})$ is negative at all points in the state space except at the origin $\mathbf{x} = 0$, where $V(\mathbf{x})$ is equal to zero.

Definition: Negative semi-definiteness: A scalar function $V(x_1, x_2, \ldots, x_n)$ is said to be negative semi-definite if $V(\mathbf{x})$ is such that $V(\mathbf{x})$ is negative at all points in the state space except at one or more points in the state space including the origin $\mathbf{x} = 0$, where it is equal to zero.

Definition: Indefiniteness: A scalar function $V(x_1, x_2, \ldots, x_n)$ is said to be indefinite if $V(\mathbf{x})$ is such that $V(\mathbf{x})$ can be positive, negative or zero anywhere in the state space.

Sign Definiteness of Scalar Functions

Example $V(\mathbf{x}) = x_1^2 + x_2^2$ is *positive definite* in the *two* dimensional ($n = 2$) state space since $V(0,0) = 0$ only when $x_1 = x_2 = 0$ and $V(\mathbf{x}) > \mathbf{0}$ at all other points.

$V(\mathbf{x}) = x_1^2 + x_2^2 + 3x_3^2 - x_3^4$ is positive definite in a region where $|x_3| < \sqrt{3}$

$V(\mathbf{x}) = x_1^2 + x_2^2$ is *positive semi-definite* in the *three* dimensional ($n = 3$) state space as this $V(\mathbf{x})$ function can be zero at the origin and also for any value of x_3 as $V(\mathbf{x})$ is independent of x_3.

$V(x) = (x_1 + x_2)^2 + x_3^2$ is *positive semi-definite* as it can be zero whenever $x_3 = 0$ and $x_1 = -x_2$. Similarly,

$V(\mathbf{x}) = -(x_1^2 + x_2^2)$ is negative definite in $n = 2$ and negative semi-definite in $n = 3$.

$V(\mathbf{x}) = (x_1x_2 - x_2^2x_1 + 5x_1^2x_2)$ is sign indefinite, so also $V(x) = x_1 + x_2 + x_3^2$.

Sign Definiteness in Quadratic Forms

1. $V(\mathbf{x})$ is in a quadratic form in the x_is, if $V(\mathbf{x})$ is in the form:

$$V(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}x_i x_j \text{ where } k_{ij}s \text{ are real constants} \quad (1)$$

The most commonly used quadratic forms can be written as:

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x}$$

where the matrix $\mathbf{Q}$ can be written as:

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}$$

Sign Definiteness in Quadratic Forms

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= q_{11}x_1^2 + q_{22}x_2^2 + \cdots + q_{nn}x_n^2 + (q_{12} + q_{21})x_1x_2 + (q_{13} + q_{31})x_1x_3 + \cdots$$

$$+ (q_{ij} + q_{ji})x_ix_j + \cdots \qquad (2)$$

Comparing (1) and (2)

$$q_{ij} = k_{ij} \text{ whenever } i = j$$

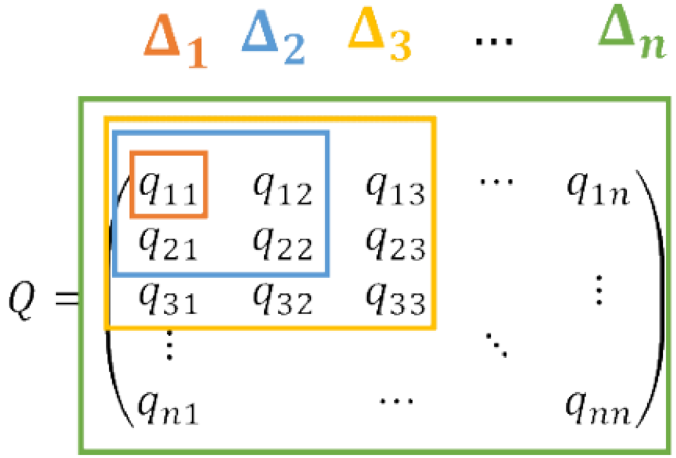
$$q_{ij} = \frac{1}{2}(k_{ij} + k_{ji}) = q_{ji} \text{ for } i \neq j$$

Sign Definiteness in Quadratic Forms

For example $V(\mathbf{x}) = 4x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + x_2x_3 + 2x_1x_3$ can be written in quadratic form as:

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Theorem Sylvester's theorem for sign definiteness of quadratic functions: A quadratic function $V(\mathbf{x}) = \mathbf{x}^T \mathbf{Q} \mathbf{x}$ is positive definite if and only if all the principal determinants $|Q_1|, |Q_2|, \ldots, |Q_n|$ of the matrix $\mathbf{Q}$ are positive.



Sign Definiteness of Scalar Functions in Quadratic Forms

Positive definiteness of scalar functions

- The condition for the scalar function to be positive definite is

By Sylvester's theorem, all the principal minors of **Q** should be positive

or

All the Eigen values of the matrix P should be greater than zero, i.e., all are positive

Positive semi definiteness of scalar functions

- The condition for the scalar function to be positive semi definite is

By Sylvester's theorem, all the principal minors of matrix **Q** is positive except the **Q** matrix and **Q** is a singular matrix

or

At least one Eigen value is zero and remaining Eigen values of the matrix **Q** are positive

Negative definiteness of scalar functions

- The condition for the scalar function to be negative definite is

All the Eigen values of the matrix P should be less than zero, i.e., all are negative.

Negative semi definiteness of scalar functions

- The condition for the scalar function to be negative semi definite is

At least one Eigen value is zero and remaining Eigen values of the matrix **Q** are negative

Indefiniteness of scalar functions

- The condition for the scalar function to be indefinite is

The Eigen values of the matrix **Q** should be a combination of positive and negative values.

Sign Definiteness of Scalar Functions in Quadratic Forms

Example Check the sign definiteness of the following quadratic forms.

$$V(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{Q}_1 = 4 > 0; \mathbf{Q}_2 = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = 7 > 0; \mathbf{Q}_3 = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{vmatrix} = 5 > 0$$

All the determinants are positive; $\mathbf{Q}$ is a positive definite matrix or $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x}$ is a positive definite function.

Stability by the Second Method of Lyapunov

Lyapunov's Stability Theorems

Lyapunov's first theorem: For a system described by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{f}(\mathbf{0}, t) = \mathbf{0}$ for all $t \geq t_0$, if there exists a scalar function $V(\mathbf{x}, t)$ having continuous first partial derivatives and satisfying the conditions,

1. $V(\mathbf{x}, t)$ is positive definite
2. $\dot{V}(\mathbf{x}, t)$ is negative definite

then the equilibrium state at the origin is uniformly asymptotically stable.

If in addition, $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, then the equilibrium state at the origin is uniformly asymptotically stable in the large.

Stability by the Second Method of Lyapunov

Lyapunov's Stability Theorems

Lyapunov's second theorem: For a system described by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{f}(\mathbf{0}, t) = 0$ for all $t \geq t_0$, if there exists a scalar function $V(\mathbf{x}, t)$ having continuous first partial derivatives and satisfying the conditions,

1. $V(\mathbf{x}, t)$ is positive definite,
2. $\dot{V}(\mathbf{x}, t)$ is negative semi-definite,
3. $\dot{V}(\mathbf{x}, t)$ does not vanish identically in $t \geq t_0$ for any t_0, along the state trajectory $\varphi(t; x_0, t_0)$,

then the equilibrium state at the origin is uniformly asymptotically stable in the large.

Stability by the Second Method of Lyapunov

Lyapunov's Instability Theorems

Instability theorem: For a system described by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

where $\mathbf{f}(\mathbf{0}, t) = 0$ for all $t \geq t_0$, if there exists a scalar function $W(\mathbf{x}, t)$ having continuous first partial derivatives and satisfying the following conditions,

1. $W(\mathbf{x}, t)$ is positive definite in some region about the origin
2. $\dot{W}(\mathbf{x}, t)$ is positive definite in the same region

Example Study the stability of the system described by:

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

Solution: The equilibrium points can be obtained by equating the derivatives to zero. i.e.

$$0 = x_2 - x_1(x_1^2 + x_2^2)$$
$$0 = -x_1 - x_2(x_1^2 + x_2^2)$$

The simultaneous solution of these yields $x_1 = x_2 = 0$, the origin is the only equilibrium point.

Consider a scalar $V(\mathbf{x}) = x_1^2 + x_2^2$ which is positive definite. The time derivative of $V(\mathbf{x})$ can be derived as:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1(x_2 - x_1(x_1^2 + x_2^2)) + 2x_2(-x_1 - x_2(x_1^2 + x_2^2)) \\ &= -2(x_1^2 + x_2^2)^2 \text{ which is negative definite.}\end{aligned}$$

The equilibrium point at the origin is asymptotically stable.

Further as $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ the origin is asymptotically stable in the large.

The locus of $V(\mathbf{x}) = x_1^2 + x_2^2 = C$ is a circle around the origin.

For values of $C = C_1, C_2, C_3$ etc. with $C_1 < C_2 < C_3 < \ldots$, the $V(\mathbf{x})$ function becomes concentric circles with progressively smaller radii.

As the radius is fixed for each of these circles, as the radii $\sqrt{(x_1^2 + x_2^2)}$ increases without bound to infinity, the function $V(\mathbf{x})$ also increases to infinity.

Example Consider the system

$$\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2)$$

Consider a function $W(\mathbf{x}) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned}\dot{W}(\mathbf{x}) &= (x_1\dot{x}_1 + x_2\dot{x}_2) = x_1(x_2 + x_1(x_1^2 + x_2^2)) + x_2(-x_1 + x_2(x_1^2 + x_2^2)) \\ &= (x_1^2 + x_2^2)^2\end{aligned}$$

which is positive definite. Since $W(\mathbf{x})$ and $\dot{W}(\mathbf{x})$ are both positive definite the origin is unstable.

In general, the procedure for determining stability by the second method of Lyapunov can be summarised as follows.

1. Assume a positive definite $V(\mathbf{x})$ function.
2. Determine the derivative $\dot{V}(\mathbf{x})$ for the above function as

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \ldots + \frac{\partial V}{\partial x_n}\dot{x}_n$$

3. Evaluate $\frac{dV}{dt}$ along the trajectory by substituting for the derivatives $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ from the state equations.
4. Check whether $\frac{dV}{dt}$ is negative definite in the region where $V(\mathbf{x})$ is positive definite. If not, check whether $\frac{dV}{dt}$ is at least negative semi-definite.
5. If $\frac{dV}{dt}$ is negative semi-definite, verify whether $\frac{dV}{dt}$ vanishes identically only at the origin, by substituting the condition for $\frac{dV}{dt}$ to vanish.

   Then, if $\frac{dV}{dt}$ vanishes only at the origin or $\frac{dV}{dt}$ is negative definite, conclude about the stability by either of the two theorems.

Stability of Linear Continuous Time Systems

Let the autonomous system be described by: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{x}$ is an n-dimensional state vector and $\mathbf{A}$ is $(n \times n)$ matrix. Assume that the matrix $\mathbf{A}$ is nonsingular. Then the only equilibrium point is the origin in the state space. Let

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$$

be chosen as a possible Lyapunov function, where $\mathbf{P}$ is real symmetric positive definite matrix. The time derivative of $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$ on the state trajectory can be derived as:

$$\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^{\mathrm{T}}\mathbf{P}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}} = (\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{P}\mathbf{x} + (\mathbf{x}^{\mathrm{T}})\mathbf{P}(\mathbf{A}\mathbf{x}) = \mathbf{x}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}$$

Since $V(\mathbf{x})$ is chosen to be positive definite, for asymptotic stability, we require $\dot{V}(\mathbf{x})$ to be negative definite or at least negative-semi definite. i.e., $\dot{V}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}$ is to be negative definite or $\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}$ has to be negative definite.

Stability of Linear Continuous System

Let $$\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{PA})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x}$$

where $\mathbf{Q} = -(\mathbf{A}^T\mathbf{P} + \mathbf{PA})$ is positive definite.

Hence for asymptotic stability, it is enough that $\mathbf{Q}$ is positive definite for $\mathbf{P}$ which is positive definite.

Instead of first specifying a positive definite matrix $\mathbf{P}$ and then checking whether $\mathbf{Q}$ is positive definite, it is convenient to specify a positive definite matrix $\mathbf{Q}$ and then check whether the matrix $\mathbf{P}$ obtained by solving the matrix equation $\mathbf{Q} = -(\mathbf{A}^T\mathbf{P} + \mathbf{PA})$ is positive definite. It is to be noted that $\mathbf{P}$ is positive definite is a necessary condition.

It is convenient to choose $\mathbf{Q} = \mathbf{I}$ for determining whether a positive definite matrix $\mathbf{P}$ exists, i.e. by choosing $\mathbf{I} = -(\mathbf{A}^T\mathbf{P} + \mathbf{PA})$.

Stability of Linear Continuous System

Example Determine the stability of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ by Lyapunov's theorem and hence determine a suitable Lyapunov function.

Solution: Let us choose a

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x} \text{ where } \mathbf{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \text{ and let } \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Substituting in the matrix equation $\mathbf{I} = -(\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A})$ we get

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2P_2 & -2P_3 \\ P_1 - 3P_2 & P_2 - 3P_3 \end{bmatrix} + \begin{bmatrix} -2P_2 & P_1 - 3P_2 \\ -2P_3 & P_2 - 3P_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -4P_2 & P_1 - 3P_2 - 2P_3 \\ P_1 - 3P_2 - 2P_3 & 2P_2 - 6P_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Equating $-4P_2 = -1$ $\qquad P_2 = \frac{1}{4}$
- $2P_2 - 6P_3 = -1$ $\qquad P_3 = \frac{1}{4}$
- $P_1 - 3P_2 - 2P_3 = 0$ $\qquad P_1 = \frac{5}{4}$

$$\mathbf{P} = \begin{bmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

which is positive definite as

$$|\mathbf{P}_1| = \frac{5}{4} > 0 \text{ and } \begin{vmatrix} \frac{5}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{vmatrix} = \left(\frac{5}{4}\right)\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) > 0$$

which means that **P** is positive definite and hence the origin is asymptotically stable. The Lyapunov function is

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x} = \frac{5}{4}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{4}x_2^2$$

Negative definiteness of scalar functions

- The condition for the scalar function to be negative definite is

By Sylvester's theorem, all principal minors of **Q** should be in the form $(-1)^j \; where \; j = 1, 2, 3 \ldots$., i.e., the first minor should be negative, second should be positive, third is negative, and so on

or

All the Eigen values of the matrix P should be less than zero, i.e., all are negative.

Negative semi definiteness of scalar functions

- The condition for the scalar function to be negative semi definite is

By Sylvester's theorem, all principal minors of **Q** should be in the form $(-1)^j where \; j = 1, 2, 3 \ldots$., except the **Q** matrix and **Q** is a singular matrix

or

At least one Eigen value is zero and remaining Eigen values of the matrix **Q** are negative

Sign Definiteness of Scalar Functions in Quadratic Forms

Indefiniteness of scalar functions

- The condition for the scalar function to be indefinite is

By Sylvester's theorem, all principal minors of **Q** should be in the form $(-1)^{j+1}$ $where\ j = 1, 2, 3 \ldots.,$ i.e., the first minor should be positive, second should be negative, third is positive, and so on

or

The Eigen values of the matrix **Q** should be a combination of positive and negative values.