Overview of Control Systems

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What are Models?

- System can be thought of as an entity that produces outputs corresponding to inputs provided to it.

- Model is a description of a system
- A system can be mathematically represented as a mapping,

$$f: u(t) \longrightarrow y(t) \; i.e., \; y(t) = f(u(t))$$

- Mathematical models are used to characterize system response

Dynamic System

- System output depends on present as well as past inputs
- System with memory

$$y(t) = f(u\ (t), u(t-1), u(t-2)\ \dots)$$

- Linear and non-linear systems
- Time invariant and time-variant systems

- Most practical systems are non-linear and time varying. However, they are typically modeled as linear and time invariant (LTI) to begin with.

What is Model Based Control?

- What is model-based systems engineering?
- The use of models in the design and analysis of systems.
- What is control?
- Make a system behave as desired.
- The use of mathematical models in the design of controllers that regulate the response of dynamic systems.
- Rule-based (Logic-based) control is widely used in practice.

What are the purpose of Models?

- Purpose of models - Analysis, Design, Control
- Consider a system (S) having an input (u) and an output (y).
- The following were proposed by Karplus (1983).

Problem	Given	Find	Purpose
Synthesis	u and y	S	Understand
Analysis	u and S	y	Predict
Instrumentation/Regulation	S and y	u	Control

Classification of Models

- Depending on the way time is represented,
  1. Continuous time models
  2. Discrete time models

- Depending on the way space is represented,
  1. Spatially heterogeneous models
  2. Spatially homogeneous models

- Depending on whether random events are accounted for,
  1. Stochastic models
  2. Deterministic models

Two Common Representations

- Transfer Function Representation (External Representation) - obtained by applying the Laplace transform to the governing linear ODE (with zero initial conditions).

- State Space Representation (Internal Representation): The governing ODE is rewritten as a set of first-order ODEs and analyzed.

Modern Control Theory Vs Conventional Control Theory

Conventional Control Theory

(Transfer function approach)

- Applicable only to linear time-invariant single-input, single-output systems
- A complex frequency-domain approach
- With zero initial conditions
- Provide no information regarding the internal state of the system.
- Not computer friendly
- Using input-output relationship or transfer function
- Variables represent physical quantities of the system and are measurable.

Modern Control Theory

State space approach

- Applicable to multiple-input, multiple-output systems, which may be linear or nonlinear, time-invariant or time-varying
- essentially time-domain approach and frequency domain approach (in certain cases such as H-infinity control)
- Any initial conditions
- It provides information regarding the internal state of the system.
- Easier for analysis using computers
- Any variables of the system can be used.
- It is not necessary that the state variables represent physical quantities of the system, but variables that do not represent physical quantities and those that are neither measurable nor observable may be chosen as state variables.

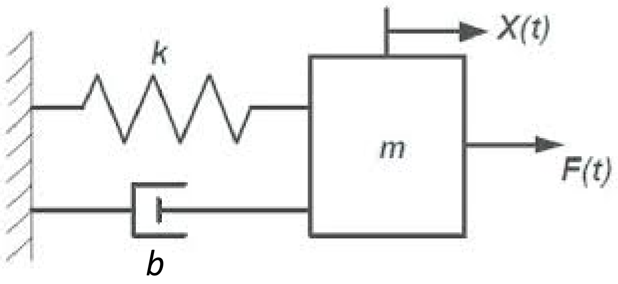
Example

Consider a mass spring damper system whose governing equation is given by,

$$m\ddot{x}(t)+b\dot{x}(t)+kx(t)=F(t)$$

The transfer function for the system can be written as

$$\frac{X(s)}{F(s)}=\frac{1}{ms^2+bs+k}$$



State Space Model

$$x_1(t)=x(t) \qquad x_2(t)=\dot{x}(t)$$

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-\frac{k}{m}x_1(t)-\frac{b}{m}x_2(t)+\frac{1}{m}u(t)$$

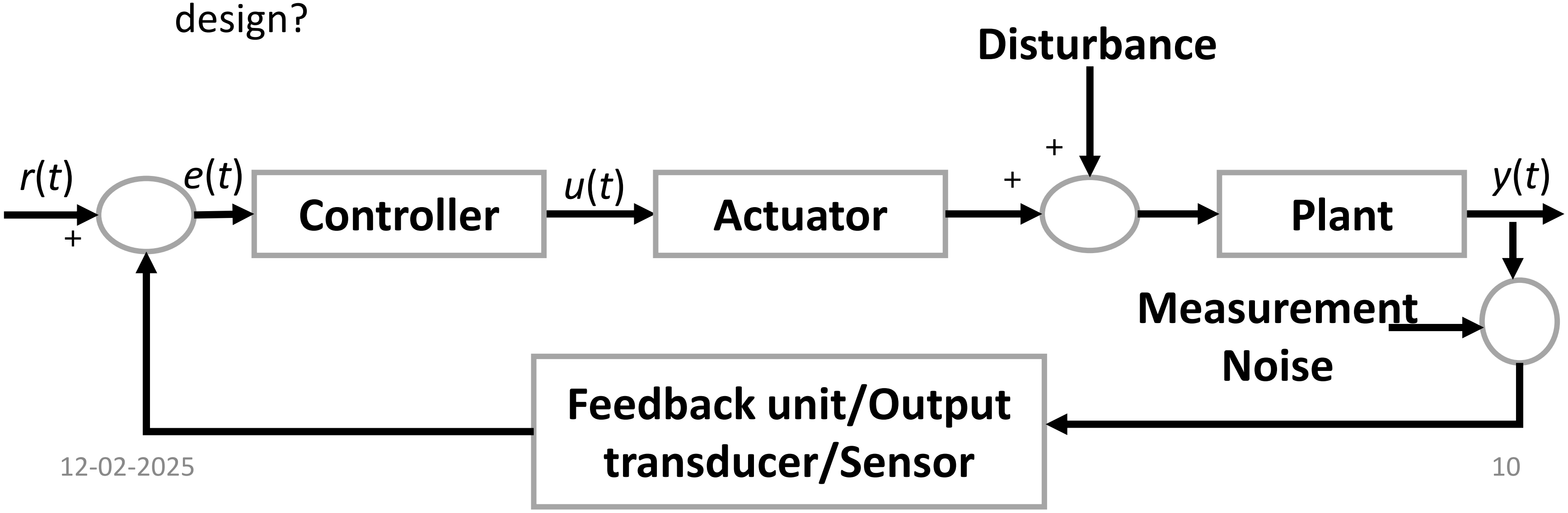
$$y(t)=x_1(t)$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -k/m & -b/m\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1/m\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Model-Based Control: “Perceive, Decide, Act”

- Model-Based Design - useful to obtain the initial range of values of controller parameters/gains that will provide **stability** and desired **performance** (a better alternative to a pure trial and error process).
- Provide insights on identifying the critical variables/parameters and their ranges, that can be subsequently used in rule-based logic.
- Questions:
  - How to develop system models?
  - How to select a controller?
  - How to choose sensors and actuators?
  - How to evaluate the importance of sensor and actuator characteristics on the design?



Review State-space Representation

- State - The state of a dynamic system is the smallest set of variables such that, knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.
- State variables
- State vector
- Input variables and output variables.
- For a linear time-invariant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \text{ - State Equation}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \text{ - Output Equation}$$

- For a linear time-invariant discrete system,

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

State Model for Continuous-Time Systems

- For a linear time-invariant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- For a linear time-variant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

- For a nonlinear time-invariant system,

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t))$$
$$\mathbf{y}(t) = \mathrm{g}(\mathbf{x}(t), \mathbf{u}(t))$$

- For a nonlinear time-variant system,

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$$
$$\mathbf{y}(t) = \mathrm{g}(\mathbf{x}(\mathrm{t}), \mathbf{u}(\mathrm{t}), t)$$

Controllability and Observability

- Consider a continuous linear time-invariant system,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Where,

$\mathbf{x}(t)$=State vector of order (n x 1)

$\mathbf{u}(t)$ =Input vector of order (r x 1)

$\mathbf{A}$=System matrix of order (n x n)

$\mathbf{B}$=Input matrix of order (n x r)

- This system is said to be completely state controllable if for any initial state $\mathbf{x}_0$ and any final state say $\mathbf{x}_1$, there exist a control input that transfers $\mathbf{x}_0$ to $\mathbf{x}_1$ in a finite time.

- This system is said to be completely state observable at time t_0 if there exists a finite time interval $[t_0, t_1]$, $t_1 > t_0$, such that the knowledge of the input $\mathbf{u}$ and the output $\mathbf{y}$ over the time interval $[t_0, t_1]$, is sufficient to uniquely determine $\mathbf{x}(t_0)$.

Kalman Filter (Discrete)

- An **estimator** is a rule for calculating an estimate of a given quantity based on observed data.

- R.E Kalman(1960) described his filter using state space technique to estimate state variables associated with stochastic processes.

- Requires information regarding the system's dynamics, statistical information of the system uncertainties/disturbances, and measurement errors.

- Kalman filter is a recursive algorithm which needs the current instant's state estimate, current input, and output measurements to calculate the next instant's state estimate. Hence, it is well suited for real time applications.

- Defining the filter in state space methods simplifies the implementation of the filter in the discrete domain.

State Space Model

State Equation

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{w}_k$$

Output Equation

$$\boldsymbol{z}_k = \mathbf{C}\mathbf{x}_k + \mathbf{v}_k$$

Where, **w** represents the process disturbance and assumed to be white noise process described by normal probability distributions with zero mean and with known covariance (**Q**),

and **v** represents the measurement noise and assumed to be white noise process described by normal probability distributions with zero mean and with known covariance (**R**).

The process disturbance and the measurement noise are assumed to be independent of each other (zero cross-correlation).

Prediction step (time update):

- The a priori state estimate is

$$\hat{\mathbf{x}}_{k+1}^{-} = \mathbf{A}\hat{\mathbf{x}}_{k}^{+} + \mathbf{B}\mathbf{u}_{k}$$

Given $\hat{\mathbf{x}}_{k}^{+}$ and $\mathbf{u}_{k}$.

- The a priori state estimation error covariance is

$$\mathbf{P}_{k+1}^{-} = \mathbf{A}\mathbf{P}_{k}^{+}\mathbf{A}^{T} + \mathbf{Q}$$

Correction step (measurement update):

- The Kalman gain is

$$\mathbf{K}_{k+1} = \mathbf{P}_{k+1}^{-}\mathbf{C}^{\mathrm{T}}\left(\mathbf{C}\,\mathbf{P}_{k+1}^{-}\mathbf{C}^{\mathrm{T}} + \mathbf{R}\right)^{-1}$$

- The a posteriori state estimate is

$$\hat{\mathbf{x}}_{k+1}^{+} = \hat{\mathbf{x}}_{k+1}^{-} + \mathbf{K}_{k+1}(\mathbf{z}_{k+1} - \mathbf{C}\,\hat{\mathbf{x}}_{k+1}^{-})$$

- The a posteriori state estimation error covariance is

$$\mathbf{P}_{k+1}^{+} = (\mathbf{I} - \mathbf{K}_{k+1}\mathbf{C})\mathbf{P}_{k+1}^{-}$$

State Feedback Controller Design - Pole Placement Technique

Consider a closed-loop control system where all state variables are available for measurement and feedback and the system considered is completely state-controllable.

- Pole placement technique - poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.
- The design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements.

State Feedback Controller Design

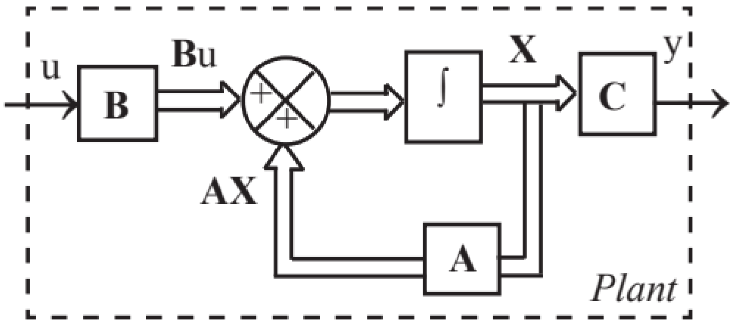


***Fig a :** System without state feedback*

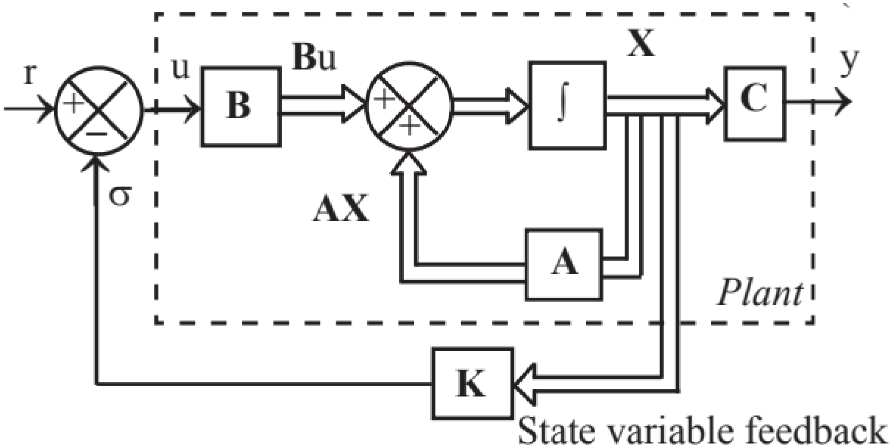


***Fig b :** System with state variable feedback*

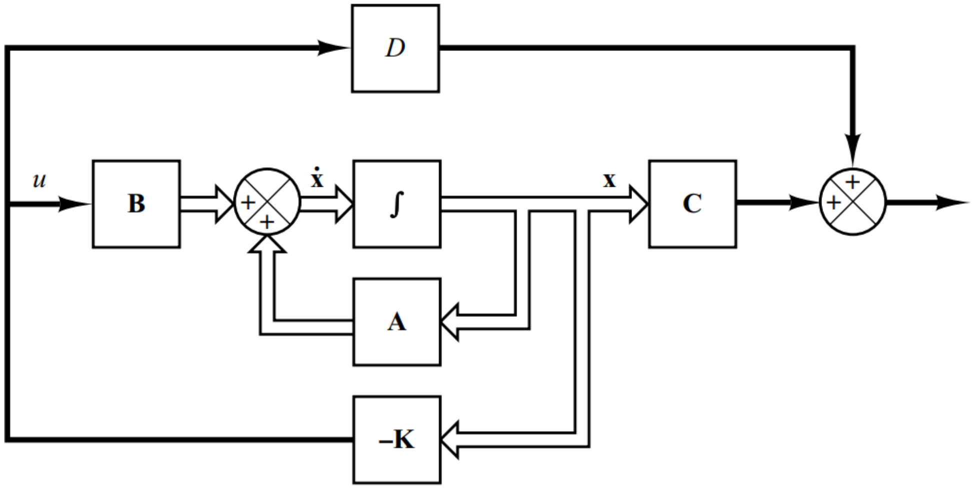
State Feedback Controller Design

Consider a control system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

where
- $\mathbf{x}$ = state vector (n-vector)
- y = output signal (scalar)
- u = control signal (scalar)
- $\mathbf{A}$ = $n \times n$ constant matrix
- $\mathbf{B}$ = $n \times 1$ constant matrix
- $\mathbf{C}$ = $1 \times n$ constant matrix
- D = constant (scalar)



We shall choose the control signal to be

$$u = -\mathbf{K}\mathbf{x}$$

State Feedback Controller Design

From $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ and $u = -\mathbf{K}\mathbf{x}$

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t)$$

The solution of this equation is given by

$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{B}\mathbf{K})t}\mathbf{x}(0)$$

where $\mathbf{x}(0)$ is the initial state caused by external disturbances.

State Feedback Controller Design

- The system matrix **A** is modified to (**A-BK**) under state feedback.
- This indicates that the eigenvalues of the matrix **A** can be modified by choosing the state feedback gain matrix **K** suitably.
- If the system is completely state-controllable, it is possible to move the eigenvalues of the original system to any desired location so that the system performance is modified according to the requirement.
- Since the eigenvalues are nothing but poles of the closed-loop transfer function, this process of placing the eigenvalues or poles of the transfer function at the desired location is called pole placement or pole assignment.

State Feedback Controller Design

- The stability and transient response characteristics are determined by the eigenvalues of matrix **A**-**BK**.
- If matrix **K** is chosen properly, the matrix **A**-**BK** can be made an asymptotically stable matrix, and for all **x**(0) ≠ **0**, it is possible to make **x**(t) approach **0** as t approaches infinity.
- The eigenvalues of matrix **A**-**BK** are called the regulator poles.
- If these regulator poles are placed in the left-half s plane, then **x**(t) approaches **0** as t approaches infinity.
- The problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

Model Predictive Control

- MPC is a model-based control strategy that optimizes an objective function and generates sequences of optimal control inputs.
- The objective function incorporates the system model with current states and the prediction of the states for a finite horizon.
- The needed minimization of the objective function associated with system performance is done by driving the states to the reference values.
- This process, including the constraints on states and control input, yields a sequence of optimal control inputs, out of which the first set is implemented.

Model-Based Control - A Case Study

- Traffic signals are the most practical means to regulate and control road traffic (congestion).
- The effectiveness of traffic signals depends on factors such as green time allocation and coordination with adjacent signals.
- Since the traffic conditions vary with time, an ideal signal must be adaptive to real-time changes in the same.

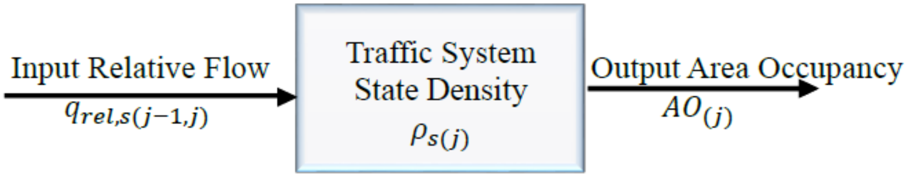
Traffic Controller Development

- Aim: To develop a macroscopic non-continuum model-based traffic control scheme for mixed traffic under over-saturated conditions.

- Objective: To maintain the traffic density of the section to a desired value (optimum density) by controlling the traffic flow.

Model Formulation-Features

- Traffic in the study stretch was characterized by traffic density, a good measure of congestion.



- Area occupancy, a variable that can capture heterogeneity and lane indiscipline, was used as measurement to estimate traffic density.

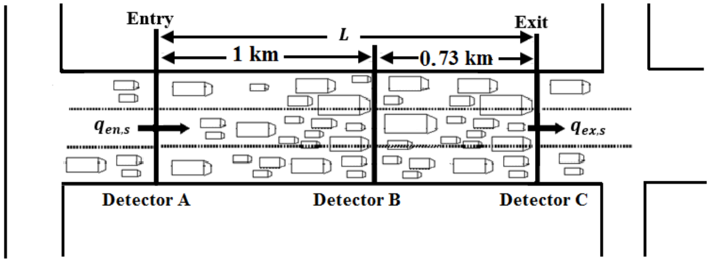


Figure: 1 Schematic of study stretch.

Model Formulation

Development of a mathematical model that describes the system using traffic variables and representation of the system in a state space form.

- **State Equation** derived from Conservation of vehicles.
- $N_{s,(j+1)} = N_{s,(j)} + h\left(q_{en,s(j,j+1)} - q_{ex,s(j,j+1)}\right).$ (1)
- Dividing by L, length of the section under study,

$$\rho_{s(j+1)} = \rho_{s(j)} + \frac{h}{L}\left(q_{en,s(j,j+1)} - q_{ex,s(j,j+1)}\right) \quad (2)$$

where h is the time period of observation.

- Relative Flow defined as, $q_{rel,s(j,j+1)} = q_{en,s(j,j+1)} - q_{ex,s(j,j+1)}$.

$$\boldsymbol{\rho_{s(j+1)} = \rho_{s(j)} + \frac{h}{L} q_{rel,s(j,j+1)}} \text{.......... State Equation} \quad (3)$$

Model Formulation…

- **Output Equation** derived from the concept of Area Occupancy

- The percent area occupancy, expressed as the proportion of time the observed vehicle occupies the chosen section of the road is given by

$$AO_{(j)} = \frac{\sum_{k=1}^{N_d} a_k \, t_{k(j)}}{Ah} 100. \qquad (4)$$

- From fundamental relation of traffic flow,

$$q_{d(j-1,j)} = \rho_{d(j)} U_{sms(j)}$$

- Density-Area occupancy relation can be derived as
-

$$AO_{(j)} = \frac{\sum_{k=1}^{N_d} a_k t_{k(j)}}{AN_d} U_{sms(j)} 100 \rho_{s(j)} \ldots\ldots\ldots\ldots \quad \text{Output Equation} \qquad (5)$$

Estimation Scheme

Goal - To estimate density from area occupancy.

- State Equation - Dynamic traffic flow equation.

  Output Equation - Density - Area occupancy relation.

- Adaptive Kalman filtering technique was used to estimate traffic density using the above equations, to incorporate the high variability in the traffic condition under consideration.

- Kalman filter was made adaptive by estimating the measurement noise statistic.

Adaptive Kalman Filter

- The residue was calculated for each time step as the difference between the actual measurement and the predicted value, given by

$$res_{(j+1)} = y_{(j+1)} - c_{(j+1)}\hat{x}^{-}_{(j+1)}. \qquad (6)$$

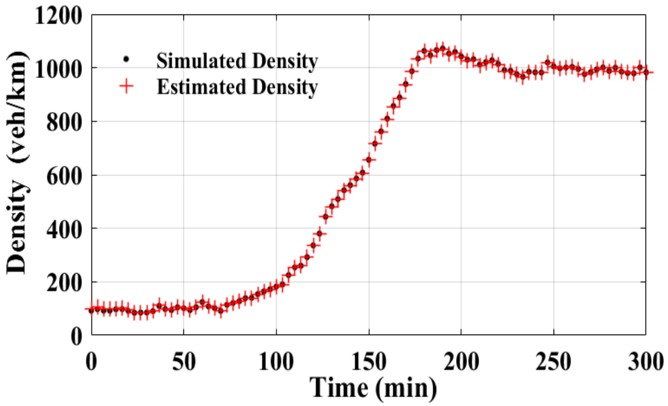
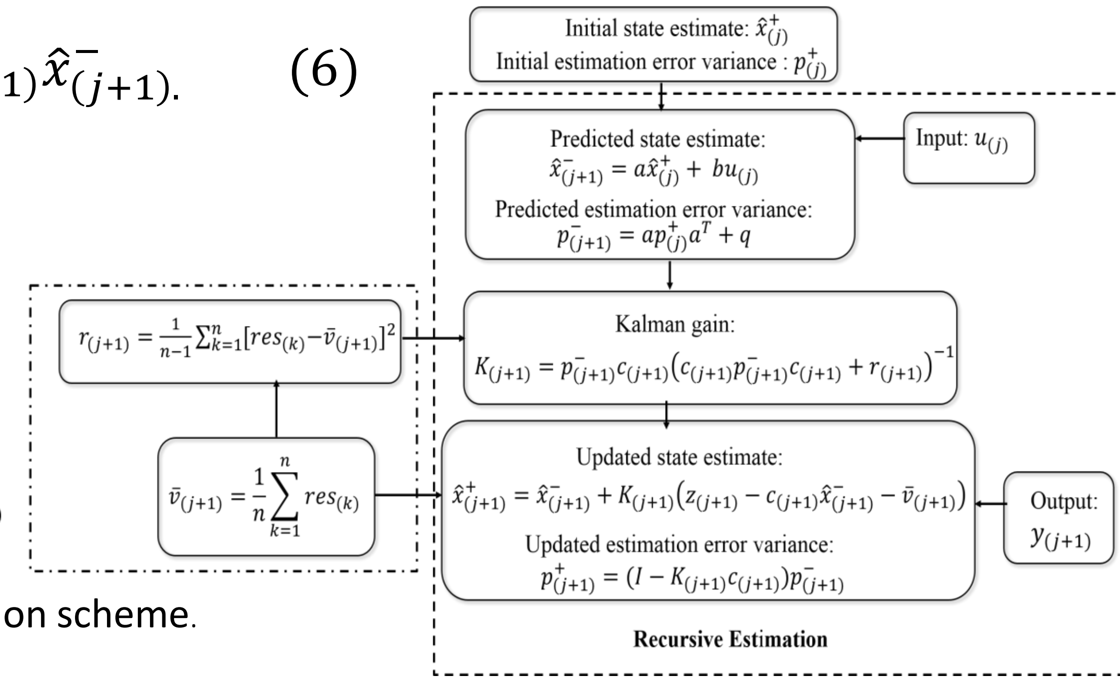


Figure 2. Corroboration of estimation scheme.



Controller Design

- The control objective of maintaining density at a desired value was attained using a Full State Feedback controller.

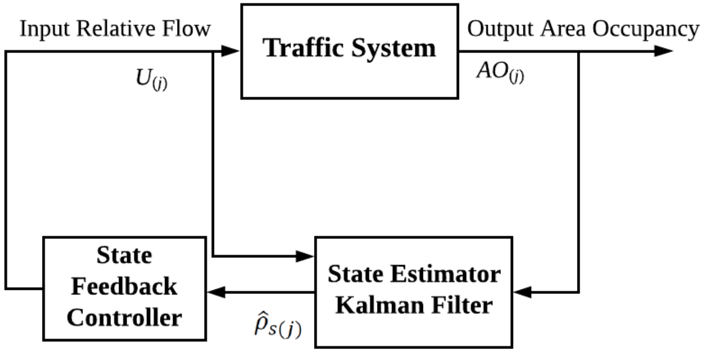


Figure 3. Schematic representation of the control scheme.

- Relative flow, which is the control input, was converted into green time using the basic concepts of saturation flow and capacity.
- Signal control regulates the amount of traffic entering and exiting the section to maintain the desired density in the study stretch.

State Feedback Controller Design

- Let the desired density to be ρ_d, the error $e_{(j)}$ at j^{th} time instant is

$$e_{(j)} = \rho_{s(j)} - \rho_d. \qquad (7)$$

- The error dynamics can be expressed as

$$e_{(j+1)} = e_{(j)} + \frac{h}{L}\left(q_{en,s(j,j+1)} - q_{ex,s(j,j+1)}\right). \qquad (8)$$

- The control equation is taken as

$$U_{(j)} = K\, e_{(j)}. \qquad (9)$$

State feedback Controller Design

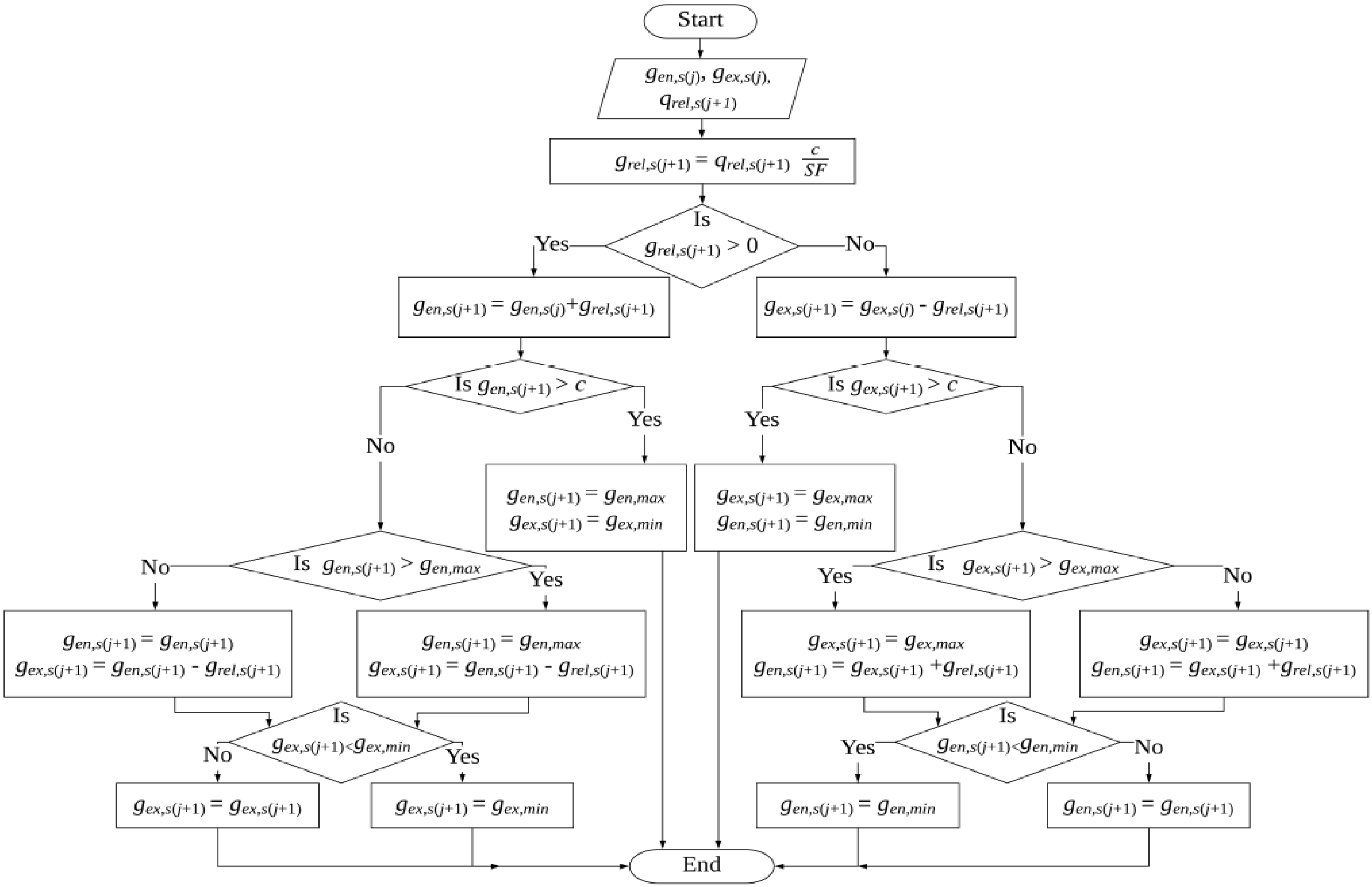
- For the system to be asymptotically stable, the closed loop pole located at

$$z = 1 + K\frac{h}{L}, \qquad (10)$$

should lie within the unit circle.

- Satisfying the above condition, the range of K was found to be (-124, 0).
- The control input $U_{(j)} = q_{rel,s,(j)}$ is converted to green times of entry and exit traffic signals using the below algorithm.

Relative flow to green time algorithm



Implementation of the Control Scheme

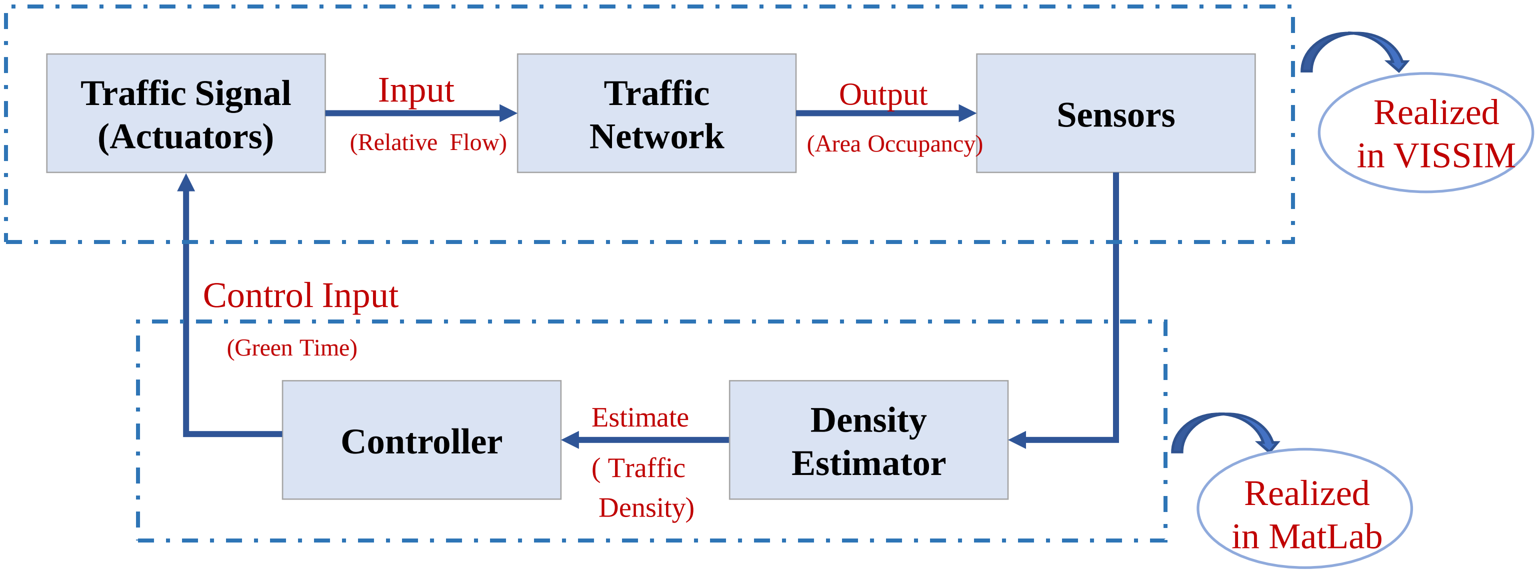


Figure 4. Schematic of the control scheme.

Results

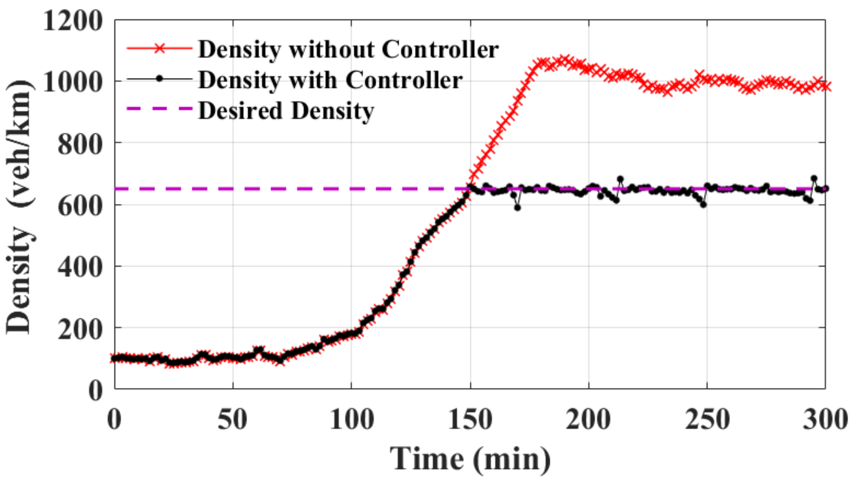


Figure 5. Density with and without controller

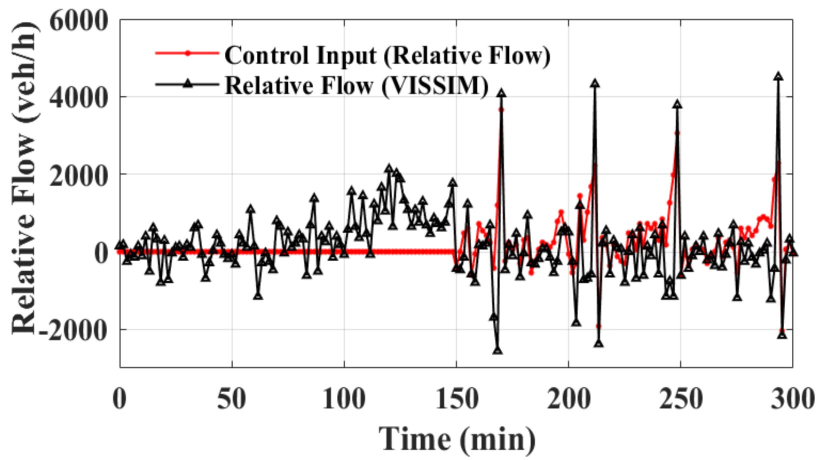


Figure 6. Control input and relative flow from VISSIM

- The developed control scheme was able to maintain the density at the desired value.
- RMS tracking error < 15 veh/km.

Results…

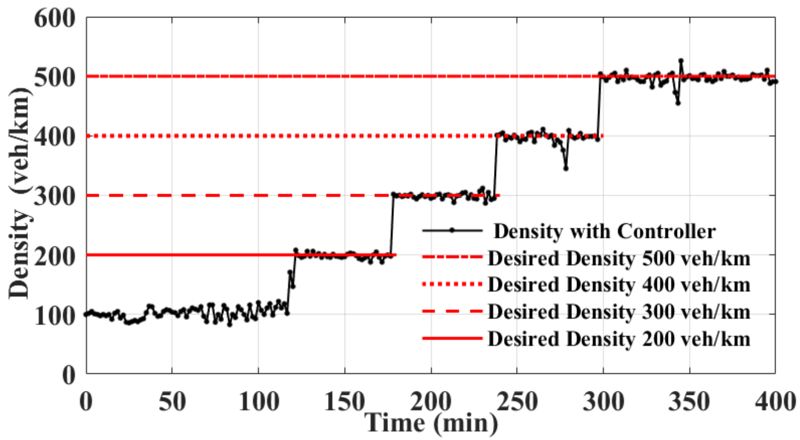


Figure 7. Density control for different desired values

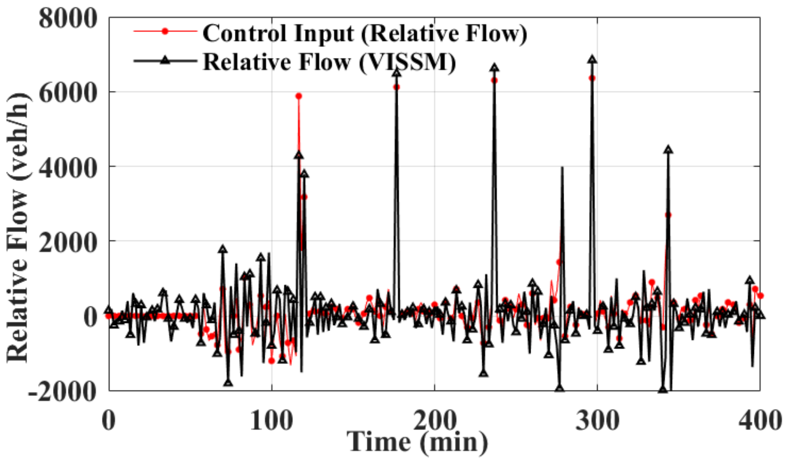


Figure 8. Relative flow from VISSIM and control input

- The controller was also able to track a time varying desired density profile.

Closing Comments

- Model-based methods are effective tools for control design

- Challenges:
  - Domain knowledge for characterizing systems
  - Knowledge of sensor and actuator characteristics
  - Effect of unmodeled dynamics
  - Variations in system parameters

Thank You